

# Mathematics 101 Midterm 2 Review Package – **Solutions**

UBC Engineering Undergraduate Society

Attempt questions to the best of your ability. This review package consists of 17 pages, including 1 cover page and 23 questions. The questions are meant to be the level of a real examination or slightly above, in order to prepare you for the real exam. Material from lectures and from the relevant textbook sections is examinable, and the problems for this package were chosen with that in mind, as well as considerations based on past examination question difficulty and style. Problems are ranked in difficulty as (\*) for easy, (\*\*) for medium, and (\*\*\*) for difficult. Note that sometimes difficulty can be subjective, so do not be discouraged if you are stuck on a (\*) problem.

There is a short table of integrals that ought to be memorized on the last page. If you have not yet committed them to memory, please refer there.

Solutions posted at: <http://ubcengineers.ca/services/academic/tutoring/>

If you believe that there is an error in these solutions, or have any questions, comments, or suggestions regarding EUS Tutoring sessions, please e-mail us at: [tutoring@ubcengineers.ca](mailto:tutoring@ubcengineers.ca). If you are interested in helping with EUS tutoring sessions in the future or other academic events run by the EUS, please e-mail [vpacademic@ubcengineers.ca](mailto:vpacademic@ubcengineers.ca).

Some of the problems in this package were not created by the EUS. Those problems originated from one of the following sources:

- Schuam's Outline of Calculus 2 ed; Ayres Jr., Frank
- Calculus – Early Transcendentals 7 ed; Stewart, James
- Calculus – 3 ed; Spivak, Michael
- Calculus Volume 1 2 ed; Apostol, Tom

All solutions prepared by the EUS.



Good Luck!

**Integrals You Should Memorize**

$$\int x^n dx = \frac{x^{n+1}}{n+1}, \quad n \neq -1$$

$$\int \frac{1}{x} dx = \log |x|$$

$$\int e^x dx = e^x$$

$$\int \sin x dx = -\cos x$$

$$\int \cos x dx = \sin x$$

$$\int \sec^2 x dx = \tan x$$

$$\int \sec x \tan x dx = \sec x$$

$$\int \frac{1}{1+x^2} dx = \arctan x$$

- (\*) 1. Compute the value of the integral.

$$\int_4^8 \frac{x}{\sqrt{x^2 - 15}} dx$$

**Solution:** For the integral

$$\int_4^8 \frac{x}{\sqrt{x^2 - 15}} dx$$

We first need to find the antiderivative

$$\int \frac{x}{\sqrt{x^2 - 15}} dx$$

Make the substitution  $u = x^2 - 15$ , and  $du = 2x dx$  to obtain

$$\begin{aligned} \int \frac{x}{\sqrt{x^2 - 15}} dx &= \frac{1}{2} \int u^{-1/2} du \\ &= \sqrt{u} \\ &= \sqrt{x^2 - 15} \end{aligned}$$

Now evaluating the antiderivative at both end points:

$$\begin{aligned} \int_4^8 \frac{x}{\sqrt{x^2 - 15}} dx &= \left( \sqrt{x^2 - 15} \right) \Big|_4^8 \\ &= 7 - 1 \\ &= 6 \end{aligned}$$

The final answer becomes

$$\int_4^8 \frac{x}{\sqrt{x^2 - 15}} dx = 6$$

- (\*) 2. If  $x = 6 \cos \theta$ ,  $y = 2 \sin \theta$ , compute the value of the integral.

$$\int_3^6 xy dx$$

**Solution:** For the integral

$$\int_3^6 xy dx$$

since  $x = 6 \cos \theta$  and  $y = 2 \sin \theta$ , we can compute  $dx = -6 \sin \theta d\theta$ . Since the bounds are from  $x = 3$

to  $x = 6$ , the equivalent  $\theta$  that accomplish this are  $\theta = \pi/3$  to  $\theta = 0$ .

$$\begin{aligned} \int_{\pi/3}^0 (6 \cos \theta)(2 \sin \theta)(-6 \sin \theta) d\theta &= -72 \int_{\pi/3}^0 \sin^2 \theta \cos \theta d\theta \\ &= 72 \frac{\sin^3 \theta}{3} \Big|_0^{\pi/3} \\ &= 24 \left( \frac{\sqrt{3}}{2} \right)^3 \\ &= 9\sqrt{3} \\ \int_3^6 xy dx &= 9\sqrt{3} \end{aligned}$$

(\*\*) 3. Evaluate the integral.

$$\int \frac{e^{2x}}{1 + e^{4x}} dx$$

**Solution:** For the integral

$$\int \frac{e^{2x}}{1 + e^{4x}} dx$$

We make the substitution  $u = e^{2x}$ , and  $du = 2e^{2x} dx$ . The integral then becomes

$$\begin{aligned} \int \frac{e^{2x}}{1 + e^{4x}} dx &= \frac{1}{2} \int \frac{1}{1 + u^2} du \\ &= \frac{1}{2} \arctan u + C \\ &= \frac{1}{2} \arctan(e^{2x}) + C \end{aligned}$$

The final answer is then

$$\int \frac{e^{2x}}{1 + e^{4x}} dx = \frac{1}{2} \arctan(e^{2x}) + C$$

(\*\*) 4. Evaluate the integral.

$$\int \frac{2x - 7}{x^2 + 1} dx$$

**Solution:** We split the integral up into the sum of two fractions, both of which are easy to evaluate:

$$\begin{aligned} \int \frac{2x - 7}{x^2 + 1} dx &= \int \frac{2x}{x^2 + 1} - \frac{7}{x^2 + 1} dx \\ &= \log(x^2 + 1) - 7 \arctan x + C \end{aligned}$$

The final answer is then

$$\int \frac{2x - 7}{x^2 + 1} dx = \log(x^2 + 1) - 7 \arctan x + C$$

(\*\*) 5. Compute the value of the integral.

$$\int_4^9 \frac{1 - \sqrt{x}}{1 + \sqrt{x}} dx$$

**Solution:**

$$\int_4^9 \frac{1 - \sqrt{x}}{1 + \sqrt{x}} dx$$

We first find an antiderivative.

$$\begin{aligned} \int \frac{1 - \sqrt{x}}{1 + \sqrt{x}} dx &= \int \frac{1 + \sqrt{x} - 2\sqrt{x}}{1 + \sqrt{x}} dx \\ &= \int 1 - \frac{2\sqrt{x}}{1 + \sqrt{x}} dx \end{aligned}$$

$$\int \frac{1 - \sqrt{x}}{1 + \sqrt{x}} dx = x - 2 \int \frac{\sqrt{x}}{1 + \sqrt{x}} dx \quad (5.1)$$

Consider the remaining integral

$$\int \frac{\sqrt{x}}{1 + \sqrt{x}} dx.$$

Make the substitution  $u = 1 + \sqrt{x}$ , which gives  $du = \frac{dx}{2\sqrt{x}}$ , or, rearranging,  $dx = 2(u - 1)du$ . We then have the integral

$$\begin{aligned} \int \frac{\sqrt{x}}{1 + \sqrt{x}} dx &= \int \frac{2(u - 1)^2}{u} du \\ &= 2 \int \frac{u^2 - 2u + 1}{u} du \\ &= 2 \int u - 2 + \frac{1}{u} du \\ &= 2 \left( \frac{u^2}{2} - 2u + \log u \right) \\ &= (1 + \sqrt{x})^2 - 4(1 + \sqrt{x}) + 2 \log(1 + \sqrt{x}) \end{aligned}$$

Then, plugging this result into equation (5.1):

$$\begin{aligned} \int \frac{1 - \sqrt{x}}{1 + \sqrt{x}} dx &= x - 2 [(1 + \sqrt{x})^2 - 4(1 + \sqrt{x}) + 2 \log(1 + \sqrt{x})] \\ &= -x + 6 + 4\sqrt{x} - 4 \log(1 + \sqrt{x}) \end{aligned}$$

Now that we have the antiderivative, we can evaluate it at the two endpoints:

$$\begin{aligned} \int_4^9 \frac{1 - \sqrt{x}}{1 + \sqrt{x}} dx &= (-x + 6 + 4\sqrt{x} - 4 \log(1 + \sqrt{x})) \Big|_4^9 \\ &= 4 \log(3/4) - 1 \end{aligned}$$

(\*\*) 6. Evaluate the integral.

$$\int \frac{e^x - 1}{e^x + 1} dx$$

**Solution:**

$$\begin{aligned} \int \frac{e^x - 1}{e^x + 1} dx &= - \int \frac{1 - e^x}{e^x + 1} dx \\ &= - \int \frac{1 + e^x}{e^x + 1} - \frac{2e^x}{e^x + 1} dx \\ &= - \int 1 - \frac{2e^x}{e^x + 1} dx \\ &= -x + 2 \int \frac{e^x}{e^x + 1} dx \\ &= -x + 2 \log(e^x + 1) + C \\ &= 2 \log(e^x + 1) - x + C \end{aligned}$$

Thus

$$\int \frac{e^x - 1}{e^x + 1} dx = 2 \log(e^x + 1) - x + C$$

(\*\*) 7. Evaluate the integral

$$\int_0^\pi e^{\cos t} \sin(2t) dt$$

**Solution:** First we will find the antiderivative. Let  $u = \cos t$ , and  $du = -\sin t dt$ .

$$\begin{aligned} \int e^{\cos t} \sin(2t) dt &= \int e^{\cos t} 2 \sin t \cos t dt \\ &= -2 \int e^u u du \\ &= -2(u - 1)e^u + C \\ &= 2(1 - \cos t)e^{\cos t} + C \end{aligned}$$

Now evaluating at the bounds of integration, we have

$$\int_0^\pi e^{\cos t} \sin(2t) dt = 2(1 - \cos \pi)e^{\cos \pi} - 0 = \frac{4}{e}$$

(\*\*) 8. Evaluate the integral

$$\int \cos(\sqrt{x}) dx$$

**Solution:** First make the substitution  $u = \sqrt{x}$ , and  $2udu = dx$ :

$$\int \cos(\sqrt{x}) dx = \int \cos(u)2udu$$

Now integrating by parts, we have

$$\int \cos(u)2udu = 2 \left( u \sin u - \int \sin u du \right)$$

Thus the final answer in terms of  $u$  is

$$2u \sin u + 2 \cos u + C$$

Then putting back in terms of  $x$ ,

$$\int \cos(\sqrt{x}) dx = 2\sqrt{x} \sin(\sqrt{x}) + 2 \cos(\sqrt{x}) + C$$

(\*\*) 9. Evaluate the integral.

$$\int \frac{1}{1 - \sin(x/2)} dx$$

**Solution:** We will need to multiply by the conjugate to solve this problem:

$$\begin{aligned} \int \frac{1}{1 - \sin(x/2)} dx &= \int \frac{1 + \sin(x/2)}{1 - \sin^2(x/2)} dx \\ &= \int \frac{1 + \sin(x/2)}{\cos^2(x/2)} dx \\ &= \int \sec^2\left(\frac{x}{2}\right) + \tan\left(\frac{x}{2}\right) \sec\left(\frac{x}{2}\right) dx \\ &= 2 \left( \tan\left(\frac{x}{2}\right) + \sec\left(\frac{x}{2}\right) \right) + C \end{aligned}$$

The final answer is then

$$\int \frac{1}{1 - \sin(x/2)} dx = 2 \left( \tan\left(\frac{x}{2}\right) + \sec\left(\frac{x}{2}\right) \right) + C$$

(\*\*\*) 10. Evaluate the integral.

$$\int \frac{\cos 2x}{\sin^2(2x) + 8} dx$$

**Solution:** For the integral

$$\int \frac{\cos 2x}{\sin^2(2x) + 8} dx$$

We make the substitution  $u = \sin 2x$ , so the differential is  $du = 2 \cos 2x dx$ .

$$\begin{aligned} \int \frac{\cos 2x}{\sin^2 2x + 8} dx &= \frac{1}{2} \int \frac{1}{u^2 + 8} du \\ &= \frac{1}{16} \int \frac{1}{\left(\frac{u}{2\sqrt{2}}\right)^2 + 1} du \end{aligned}$$

Now with this integral, make the substitution  $w = \frac{u}{2\sqrt{2}}$ , and  $du = 2\sqrt{2}dw$ , to easily evaluate the integral:

$$\begin{aligned} \frac{1}{16} \int \frac{1}{\left(\frac{u}{2\sqrt{2}}\right)^2 + 1} du &= \frac{\sqrt{2}}{8} \int \frac{1}{1 + w^2} dw \\ &= \frac{\sqrt{2}}{8} \arctan w + C \\ &= \frac{\sqrt{2}}{8} \arctan \left( \frac{\sin 2x}{2\sqrt{2}} \right) + C \end{aligned}$$

(\*\*\*) 11. Compute the value of the integral.

$$\int_0^{\pi} \frac{\sin x}{\cos^2 x - 6 \cos x + 13} dx$$

**Solution:**

$$\int_0^{\pi} \frac{\sin x}{\cos^2 x - 6 \cos x + 13} dx$$

First we find an antiderivative by making a  $u$  substitution. For the integral

$$\int \frac{\sin x}{\cos^2 x - 6 \cos x + 13} dx$$

Make the substitution  $u = \cos x$ , and  $du = -\sin x dx$ . This yields the integral

$$\begin{aligned} \int \frac{\sin x}{\cos^2 x - 6 \cos x + 13} dx &= - \int \frac{1}{u^2 - 6u + 13} du \\ &= - \int \frac{1}{(u - 3)^2 + 4} du \\ &= -\frac{1}{4} \int \frac{1}{\left(\frac{u-3}{2}\right)^2 + 1} du \end{aligned}$$

Now with this integral, we make the substitution  $\frac{u-3}{2} = w$  and  $2dw = du$ . Then we have the integral

$$\begin{aligned} -\frac{1}{4} \int \frac{1}{\left(\frac{u-3}{2}\right)^2 + 1} du &= -\frac{1}{2} \int \frac{1}{1 + w^2} dw \\ &= -\frac{1}{2} \arctan w \end{aligned}$$



Now substitution in the original variables:

$$-\frac{1}{2} \arctan\left(\frac{\cos x - 3}{2}\right)$$

Now we evaluate at the endpoints:

$$\left(-\frac{1}{2} \arctan\left(\frac{\cos x - 3}{2}\right)\right)\Big|_0^\pi$$

This yields the final answer:

$$\frac{1}{2} \left(\arctan 2 - \frac{\pi}{4}\right)$$

(\*\*\*) 12. Evaluate the integral

$$\int x \arctan x dx$$

**Solution:** First we need to compute

$$\int \arctan x dx$$

The way to do this is to integrate by parts, letting  $u' = 1$ , and  $v = \arctan x$ . Then we have

$$\int \arctan x dx = x \arctan x - \int \frac{x}{x^2 + 1} dx$$

This then evaluates to

$$\int \arctan x dx = x \arctan x - \frac{1}{2} \log(1 + x^2)$$

Now we return to the original integral. We will have to integrate by parts. Let  $x = u'$ , and  $\arctan x = v$ . Then we have

$$\begin{aligned} \int x \arctan x dx &= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2}{1 + x^2} dx \\ &= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2 + 1}{1 + x^2} - \frac{1}{1 + x^2} dx \\ &= \frac{x^2}{2} \arctan x - \frac{x}{2} + \frac{\arctan x}{2} + C \end{aligned}$$

Thus we have

$$\int x \arctan x dx = \frac{x^2}{2} \arctan x - \frac{x}{2} + \frac{\arctan x}{2} + C$$

(\*\*) 13. Evaluate the integral

$$\int \cos(\log x) dx$$

**Solution:** First, let  $\log x = u$ . Then  $du = dx/x = e^{-u}dx$ . Thus  $dx = e^u du$ . We can then transform the integral as

$$\int \cos(\log x) dx = \int \cos u e^u du$$

We then integrate by parts twice, both times integrating the exponential, and differentiating the trigonometric function. Note that we would get the same result had we instead differentiated the exponential, and integrated the trigonometric function.

$$\int e^u \cos u du = e^u \cos u + e^u \sin u - \int e^u \cos u du$$

Rearranging, we have

$$2 \int e^u \cos u du = e^u (\cos u + \sin u)$$

Then solving for the integral, we have

$$\int e^u \cos u du = \frac{e^u}{2} (\cos u + \sin u)$$

Now plugging back in the original variables, we have

$$\int \cos(\log x) dx = \frac{x}{2} (\cos \log x + \sin \log x) + C$$

(\*\*) 14. Evaluate the integral.

$$\int (\sin x)(\sin 3x) dx$$

**Solution:** We will have to integrate this by parts twice:

$$\begin{aligned} \int (\sin x)(\sin 3x) dx &= -\cos x \sin 3x + 3 \int \cos x \cos 3x dx \\ &= -\cos x \sin 3x + 3 \left( \sin x \cos 3x + 3 \int \sin x \sin 3x dx \right) \\ &= -\cos x \sin 3x + 3 \sin x \cos 3x + 9 \int \sin x \sin 3x dx \end{aligned}$$

Rearranging:

$$\begin{aligned} -8 \int \sin x \sin 3x dx &= -\cos x \sin 3x + 3 \sin x \cos 3x \\ \int \sin x \sin 3x dx &= \frac{1}{8} \cos x \sin 3x - \frac{3}{8} \sin x \cos 3x + C \end{aligned}$$

(\*\*\*) 15. Evaluate the integral.

$$\int \sqrt{1 - \cos x} dx$$

**Solution:** Use the double angle identity

$$1 - \cos 2x = 2 \sin^2 x \Rightarrow 1 - \cos x = 2 \sin^2 \left( \frac{x}{2} \right)$$

$$\begin{aligned} \int \sqrt{1 - \cos x} dx &= \int \sqrt{2 \sin^2 \left( \frac{x}{2} \right)} dx \\ &= \sqrt{2} \int \sin \left( \frac{x}{2} \right) dx \\ &= -2\sqrt{2} \cos \left( \frac{x}{2} \right) + C \end{aligned}$$

$$\int \sqrt{1 - \cos x} dx = -2\sqrt{2} \cos \left( \frac{x}{2} \right) + C$$

- (\*) 16. Find the volume of the solid generated by revolving the plane area bounded by  $x - y - 7 = 0$ ,  $x = 9 - y^2$  about the  $y$ -axis.

**Solution:**

$$\begin{aligned} V &= \int_{-2}^1 \pi x_{\text{outer}}^2 - \pi x_{\text{inner}}^2 dy \\ &= \pi \int_{-2}^1 (9 - y^2)^2 - (y + 7)^2 dy \\ &= \frac{333\pi}{5} \end{aligned}$$

- (\*) 17. Find the volume of the solid generated by revolving the plane area bounded by  $y = x^3$ ,  $y = 0$ ,  $x = 2$ , about the line  $x = 2$ .

**Solution:** Set up the integral with the radius of each disk as  $2 - x$ .

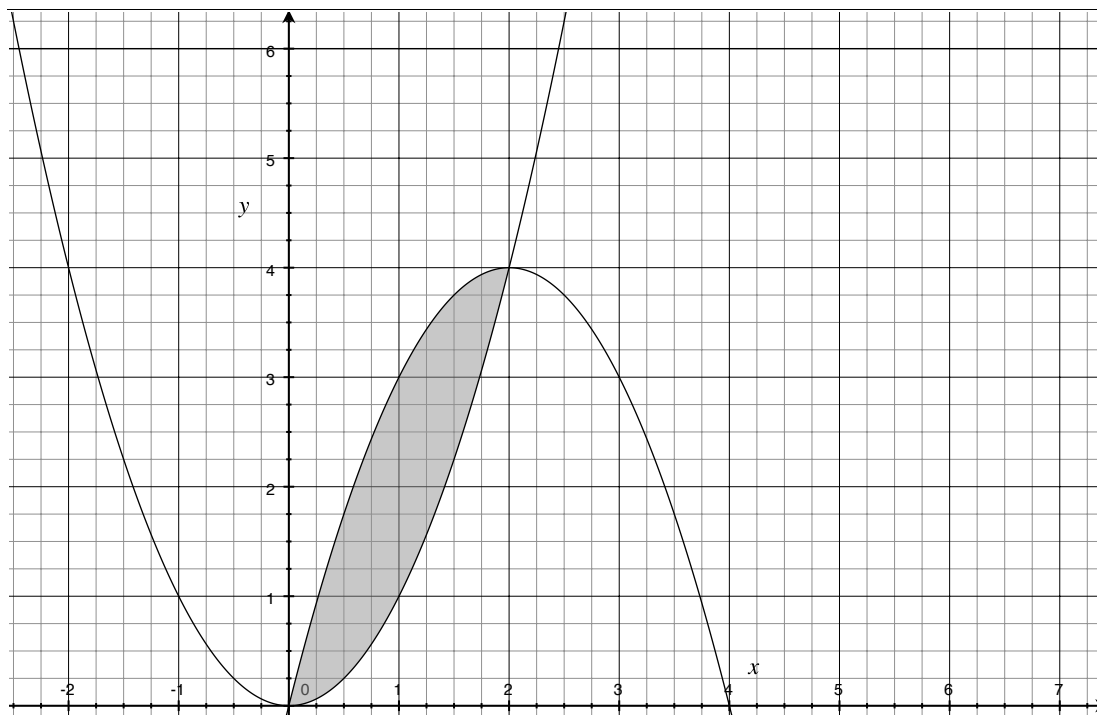
$$\begin{aligned} V &= \int_0^8 (2 - x)^2 \pi dy \\ &= \int_0^8 (2 - y^{1/3})^2 \pi dy \\ &= \frac{16\pi}{5} \end{aligned}$$

- (\*) 18. Find the volume of the solid generated by revolving the plane area bounded by  $y = x^2$  and  $y = 4x - x^2$  around the line  $y = 6$ .

**Solution:** Since we are revolving about the line  $y = 6$ , we will have to increase the radius of the disk in question.

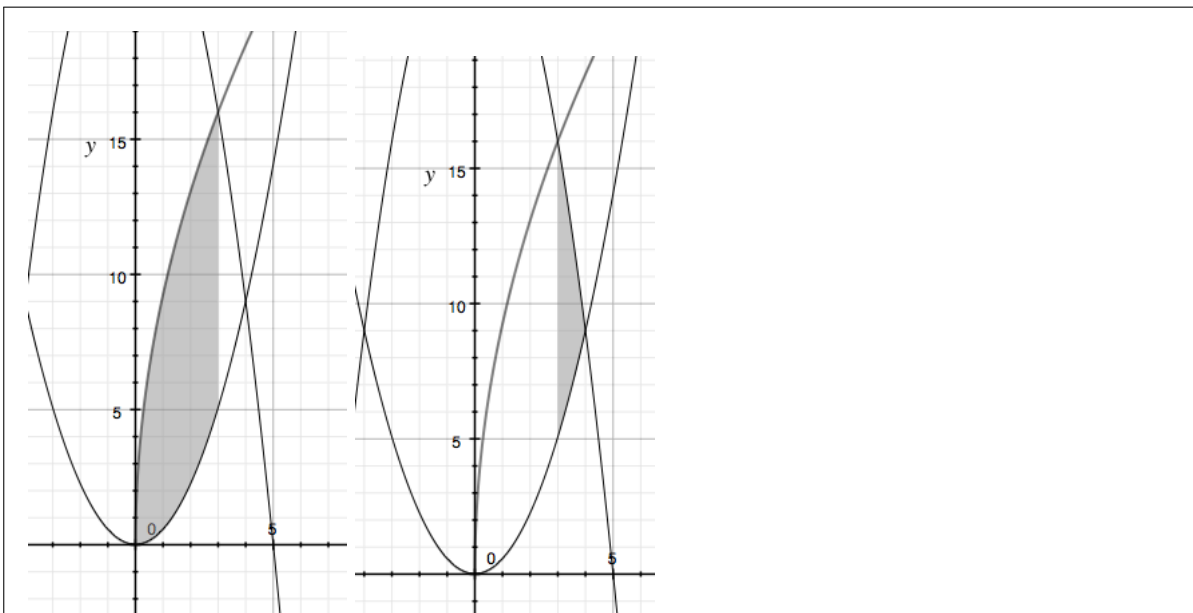
$$\begin{aligned}
 V &= \int_0^2 \pi(6 - y_{\text{outer}})^2 - \pi(6 - y_{\text{inner}})^2 dx \\
 &= \pi \int_0^2 12(y_{\text{inner}} - y_{\text{outer}}) + y_{\text{outer}}^2 - y_{\text{inner}}^2 dx \\
 &= \pi \int_0^2 12(4x - 2x^2) + x^4 - (4x - x^2)^2 dx \\
 &= \frac{64\pi}{3}
 \end{aligned}$$

Please see the figure for an illustration of the region being rotated.



(\*\*) 19. Compute the area enclosed by the curves:  $y = 25 - x^2$ ,  $256x = 3y^2$ ,  $16y = 9x^2$

**Solution:** See the following two figures which represent the two areas we need to sum:



First we find out where the curves intersect, which gives us two distinct regions separated at  $x = 3$ . Thus the area will be the sum of the two shaded regions shown in the figures.

$$A_1 = \int_0^3 \left[ \left( \frac{256}{3}x \right)^{1/2} - \frac{9}{16}x^2 \right] dx$$

$$A_2 = \int_3^4 \left[ 25 - x^2 - \frac{9}{16}x^2 \right] dx$$

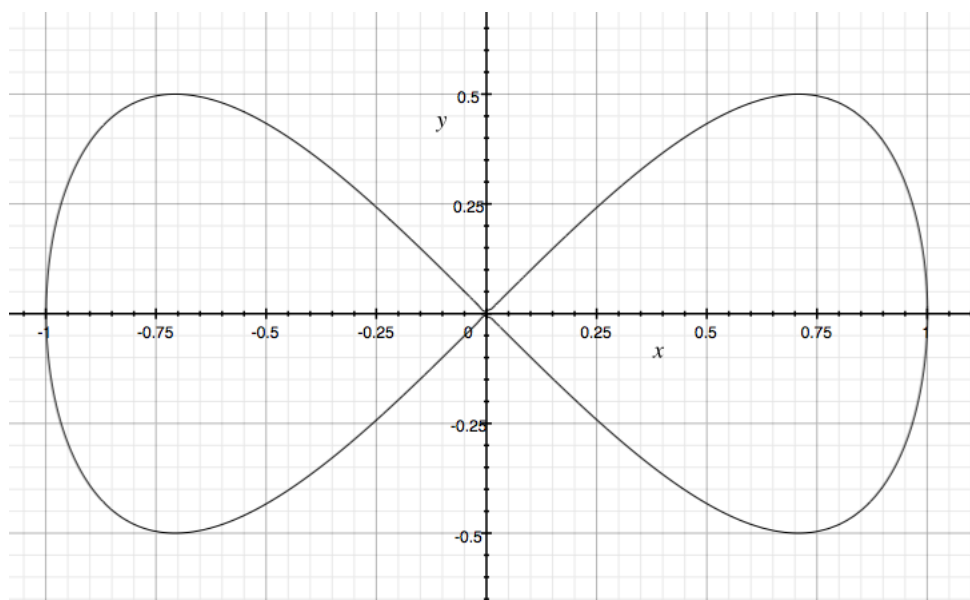
The two integrals are easily evaluated, and the sum of the two is then

$$A_1 + A_2 = \frac{98}{3}$$

(\*\*\*) 20. Compute the area enclosed by the curve  $y^2 = x^2 - x^4$ .

**Hint.** This curve is symmetric with respect to the  $x$  axis and the  $y$  axis. How can you use symmetry to help you calculate the area?

**Solution:** See the figure which shows the area we need to compute.



First we make some observations as to how one would sketch the graph.

- First note that this graph is symmetric with respect to both the  $x$  axis and  $y$  axis, because replacing  $x$  by  $-x$  or  $y$  by  $-y$  leaves the relation unchanged. This means we can find the area enclosed in one quadrant, then multiply by 4.
- There are zeros at  $x = 0$  and  $x = 1$ , which means we can draw an arc between those two points, then reflect it about the  $x$  and  $y$  axes for the total graph.
- The graph cannot extend beyond  $x = 1$  because otherwise the expression inside square root that is obtained when solving for  $y$  would become negative. Solving for  $y$ ,

$$y = x\sqrt{1-x^2}$$

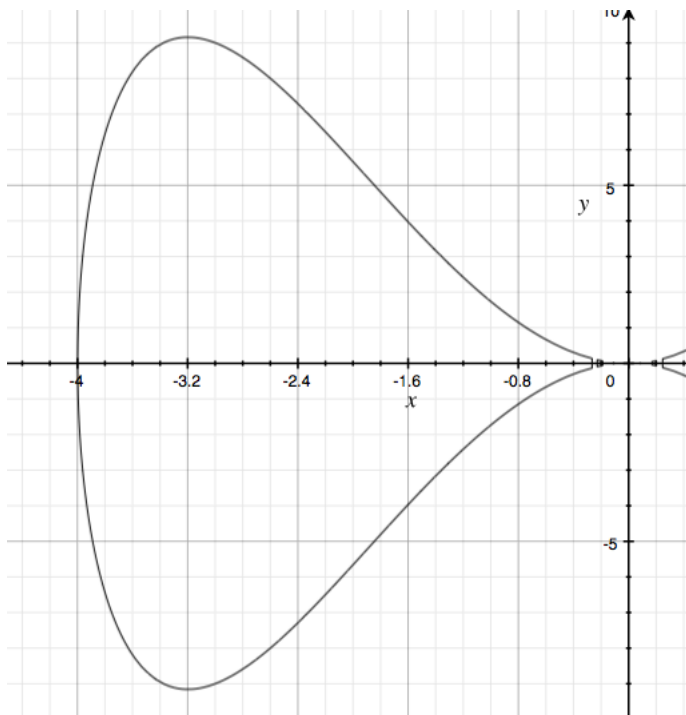
which holds for the portion in the first quadrant.

Integrating this expression from 0 to 1, then multiplying by 4 yields

$$4 \int_0^1 x\sqrt{1-x^2} dx = 4 \left( \frac{-1}{3} \right) (1-x^2)^{3/2} \Big|_0^1 = \frac{4}{3}$$

- (\*\*\*) 21. Compute the area enclosed by the curve  $y^2 = x^4(x+4)$  for the portion of the graph lying to the left of the  $y$ -axis.

**Solution:** See the figure which shows the area we need to compute.



First we need to determine how this graph is constructed:

- The graph is symmetric with respect to the  $x$  axis because replacing  $y$  by  $-y$  leaves the relation unchanged.
- There are zeros at  $x = 0$  and  $x = -4$
- The graph does not continue past  $x = -4$  because that would make the inside of the square root negative when solving for  $y$ .

$$y = x^2\sqrt{x+4}$$

This means that we draw a curve going through  $x = 0$  and  $x = -4$  that is also vertically symmetric. The expression for the area is given by

$$2 \int_{-4}^0 x^2\sqrt{x+4}dx$$

First we must find the antiderivative. To evaluate this integral, let  $x + 4 = u^2$  and thus  $dx = 2udu$ . We then obtain the integral

$$\begin{aligned} \int x^2\sqrt{x+4}dx &= 2 \int (u^2 - 4)^2udu \\ &= 2 \int u^5 - 8u^3 + 16udu \\ &= 2 \left( \frac{u^6}{6} - 2u^4 + 8u^2 \right) \\ &= \frac{(x+4)^3}{3} - 4(x+4)^2 + 16(x+4) \end{aligned}$$

Now plugging in the bounds of the original integral to find the area:

$$\begin{aligned} 2 \int_{-4}^0 x^2 \sqrt{x+4} dx &= 2 \left( \frac{(x+4)^3}{3} - 4(x+4)^2 + 16(x+4) \right) \Big|_{-4}^0 \\ &= \frac{128}{3} \end{aligned}$$

- (\*\*\*) 22. A solid has a base in the form of an ellipse with major diameter 10 and minor diameter 8. Find the volume if every section perpendicular to the major axis is an isosceles triangle with altitude 6.

**Solution:** We will have to determine the area of the cross sectional triangle  $A(x)$  as a function of the coordinate  $x$  along the ellipse. The width of the rectangle at position  $x$  will be  $2y$ , and the height will be 6, so the area is then  $A(x) = (2y)(6)/2$  because it is an isosceles triangle. Thus we have

$$\begin{aligned} V &= \int_{-5}^5 A(x) dx \\ &= \int_{-5}^5 6y dx \\ &= \int_{-5}^5 6 \left( \frac{4}{5} \sqrt{25-x^2} \right) dx \\ &= \frac{24}{5} \int_{-5}^5 \sqrt{25-x^2} dx \\ &= \left( \frac{24}{5} \right) \left( \frac{5^2 \pi}{2} \right) \\ &= 60\pi \end{aligned}$$

- (\*\*\*) 23. The base of a solid is the circle  $x^2 + y^2 = 16x$ , and every plane section perpendicular to the  $x$ -axis is a rectangle whose height is twice the distance of the plane of the section from the origin. Find the volume of this solid.

**Solution:** We will have to determine the area of the cross sectional rectangle  $A(x)$  as a function of the coordinate  $x$  along the circle. The circle  $x^2 + y^2 = 16x$  can be rewritten  $(x-8)^2 + y^2 = 64$ , by completing the square. Thus it is a circle of radius 8 centred at  $(x, y) = (8, 0)$ . The base of the rectangle will have length  $2y$ , and the height of the rectangle will be  $2x$ , so the area is then



$$A(x) = (2x)(2y) = 4xy.$$

$$\begin{aligned} V &= \int_0^{16} A(x) dx \\ &= \int_0^{16} 4xy dx \\ &= \int_0^{16} 4x\sqrt{16x - x^2} dx \\ &= \int_0^{16} 4x\sqrt{16x - x^2} dx \\ &= \int_0^{16} 4x\sqrt{64 - (8 - x)^2} dx \end{aligned}$$

To evaluate the antiderivative, make the substitution  $8 - x = u$ , and  $-du = dx$  to obtain

$$\begin{aligned} \int_0^{16} 4x\sqrt{64 - (8 - x)^2} dx &= 4 \int_{-8}^8 (8 - u)\sqrt{64 - u^2} du \\ &= 4 \left[ 8 \int_{-8}^8 \sqrt{64 - u^2} du - \int_{-8}^8 u\sqrt{64 - u^2} du \right] \\ &= 4 \left[ 8 \frac{8^2\pi}{2} + \frac{1}{3} (64 - u^2)^{3/2} \Big|_{-8}^8 \right] \\ &= 1024\pi \end{aligned}$$