Mathematics 101 Quiz 4 Review Package – **Solution**

UBC Engineering Undergraduate Society

Attempt questions to the best of your ability. This review package consists of 13 pages, including 1 cover page and 20 questions. The questions are meant to be the level of a real examination or slightly above, in order to prepare you for the real exam. Material from lectures and from the relevant textbook sections is examinable, and the problems for this package were chosen with that in mind, as well as considerations based on past examination question difficulty and style. Problems are ranked in difficulty as $(*)$ for easy, $(**)$ for medium, and $(***)$ for difficult. Note that sometimes difficulty can be subjective, so do not be discouraged if you are stuck on a $(*)$ problem.

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Some of the problems in this package were not created by the EUS. Those problems originated from one of the following sources:

- Schuam's Outline of Calculus 2 ed; Ayres Jr., Frank
- Calculus – Early Transcendentals 7 ed; Stewart, James
- Calculus – 3 ed; Spivak, Michael
- Calculus Volume 1 2 ed; Apostol, Tom

All solutions prepared by the EUS.

Good Luck!

(∗) 1. Find the average value of $f(x)$ on the interval $[4, 8]$.

$$f(x) = \frac{x}{\sqrt{x^2 - 15}}$$

Solution: We apply the formula for the average value of a function on an interval.

$$\begin{aligned} \overline{f} &= \frac{1}{b-a}\int_a^b f(x)dx \\ &= \frac{1}{4}\int_4^8 \frac{x}{\sqrt{x^2-15}}dx \\ &= \frac{6}{4} \\ &= \frac{3}{2} \end{aligned}$$

(∗) 2. Find the average value of $f(x)$ on the interval $[3, 4]$.

$$f(x) = \frac{1}{25 - x^2}$$

Solution: We apply the formula for the average value of a function on an interval.

$$\begin{aligned} \overline{f} &= \frac{1}{b-a}\int_a^b f(x)dx \\ &= \int_3^4 \frac{1}{25-x^2}dx \\ &= \frac{1}{5}\log\left(\frac{3}{2}\right) \end{aligned}$$

(∗) 3. If the following improper integral is convergent, evaluate it. Otherwise show that it is divergent.

$$\int_0^\infty \frac{dx}{x^2+4}$$

Solution: The upper limit of the integral is infinite, so we need to take a limit.

$$\begin{aligned} \int_0^\infty \frac{dx}{x^2+4} &= \lim_{k\to\infty}\int_0^k \frac{dx}{x^2+4} \\ &= \lim_{k\to\infty}\frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big|_0^k \\ &= \frac{1}{2}\lim_{k\to\infty}\arctan\left(\frac{k}{2}\right) \\ &= \frac{\pi}{4} \end{aligned}$$

(**) 4. Find the average value of $f(x)$ on the interval $(2,5)$.

$$f(x) = \frac{1}{\sqrt{|x-3|}}$$

Solution: We apply the formula for the average value of a function on an interval, noting that there is a vertical asymptote at $x = 3$. This means we will have to split up the integral.

$$\begin{aligned}
\overline{f} &= \frac{1}{b-a}\int_a^b f(x)dx \\
&= \frac{1}{3}\left(\int_2^3 \frac{dx}{\sqrt{3-x}} + \int_3^5 \frac{dx}{\sqrt{x-3}}\right) \\
&= \frac{1}{3}\left(\lim_{k\to 3^-}\int_2^k \frac{dx}{\sqrt{3-x}} + \lim_{l\to 3^+}\int_l^5 \frac{dx}{\sqrt{x-3}}\right) \\
&= \frac{1}{3}\left(\lim_{k\to 3^-}\left(-2\sqrt{3-x}\right)\Big|_2^k + \lim_{l\to 3^+}\left(2\sqrt{x-3}\right)\Big|_l^5\right) \\
&= \frac{1}{3}\left(\lim_{k\to 3^-} 2\left(-\sqrt{3-k}+\sqrt{1}\right) + \lim_{l\to 3^+} 2\left(\sqrt{5-3}-\sqrt{l-3}\right)\right) \\
&= \frac{2}{3}\left(1+\sqrt{2}\right)
\end{aligned}$$

(**) 5. Find the centroid of the region bounded by the curves $y = 4x - x^2$, $y = x$.

Solution: The curves intersect at $x = 3$, and in this interval $(0,3)$, the curve $y = 4x - x^2$ is going to be above the curve $y = x$. Thus the area of the region is

$$A = \int_0^3 (4x - x^2 - x)dx = \frac{9}{2}$$

Now we can calculate the x and y coordinates of the centroid.

$$\begin{aligned}
\overline{x} &= \frac{1}{A}\int_0^3 x(f(x)-g(x))dx \\
&= \frac{1}{A}\int_0^3 x(4x-x^2-x)dx \\
&= \frac{1}{A}\int_0^3 x(3x-x^2)dx \\
&= \frac{1}{A}\int_0^3 3x^2 - x^3 dx \\
&= \frac{27}{4}\cdot\frac{2}{9} \\
&= \frac{3}{2}
\end{aligned}$$

Now the y coordinate:

$$\begin{aligned}\overline{y} &= \frac{1}{2A}\int_0^3 (f(x))^2 - (g(x))^2 dx \\ &= \frac{1}{2A}\int_0^3 (4x - x^2)^2 - x^2 dx \\ &= \frac{108}{5}\cdot\frac{2}{9}\cdot\frac{1}{2} \\ &= \frac{12}{5}\end{aligned}$$

Thus the centroid is

$$(\overline{x}, \overline{y}) = \left(\frac{3}{2}, \frac{12}{5}\right)$$

(∗∗) 6. If the following improper integral is convergent, evaluate it. Otherwise show that it is divergent.

$$\int_0^3 \frac{dx}{\sqrt{9-x^2}}$$

Solution: We note that the integrand has a singularity at $x = 3$, so we will have to take a limit (from the left) at the upper bound of integration.

$$\begin{aligned}\int_0^3 \frac{dx}{\sqrt{9-x^2}} &= \lim_{k\to 3^-}\int_0^k \frac{dx}{\sqrt{9-x^2}} \\ &= \lim_{k\to 3^-} \arcsin\left(\frac{x}{3}\right)\Big|_0^k \\ &= \lim_{k\to 3^-} \arcsin\left(\frac{k}{3}\right) \\ &= \frac{\pi}{2}\end{aligned}$$

(∗∗) 7. Show that the integral converges.

$$\int_1^\infty e^{-x^2} dx$$

Solution: First we need to establish a relevant inequality about the integrand. For $x \in (1, \infty)$, we have

$$\begin{aligned} 0 < x &\leq x^2 \\ -x &\geq -x^2 \\ e^{-x} &\geq e^{-x^2} \\ \int_1^\infty e^{-x^2} dx &\leq \int_1^\infty e^{-x} dx \end{aligned}$$

Now we show that e^{-x} has finite area underneath it from 1 to ∞.

$$\begin{aligned} \int_1^\infty e^{-x} dx &= \lim_{n\to\infty} \int_1^n e^{-x} dx \\ &= -\lim_{n\to\infty} e^{-x}\Big|_1^n \\ &= \frac{1}{e} \\ &< \infty \end{aligned}$$

Therefore, by the comparison, since both e^{-x} and e^{-x^2} are positive, and e^{-x} is convergent the integral

$$\int_1^\infty e^{-x^2} dx$$

also converges.

(**) 8. Determine if the integral converges or diverges.

$$\int_1^\infty \frac{1}{\sqrt{x^4+2x+6}} dx$$

Solution: First we see that, for large x, the integrand looks like $1/x^2$. This suggests that it will converge. Thus we establish an inequality that the integrand satisfies. For $x \in (1, \infty)$,

$$0 \leq \frac{1}{\sqrt{x^4+2x+6}} \leq \frac{1}{\sqrt{x^4}} = \frac{1}{x^2}$$

Since both functions are positive, we can conclude that

$$\int_1^\infty \frac{dx}{\sqrt{x^4+2x+6}} \leq \int_1^\infty \frac{dx}{x^2}$$

Since

$$\begin{aligned} \int_1^\infty \frac{1}{x^2} dx &= \lim_{n\to\infty} \int_1^n \frac{1}{x^2} dx \\ &= \lim_{n\to\infty} \left(\frac{-1}{n} + 1 \right) \\ &= 1 \\ &< \infty \end{aligned}$$

By the comparison test, the second integral also converges.

(∗∗) 9. Determine if the integral converges or diverges.

$$\int_0^\infty \frac{x}{\sqrt{x^4+1}}dx$$

Solution: For large x, the integral looks like $1/x$, so this suggests it will diverge. The divergence or convergence of this integral is determined by its behaviour as x becomes large. In particular if we rewrite

$$\int_0^\infty \frac{x}{\sqrt{x^4+1}}dx = \int_0^1 \frac{x}{\sqrt{x^4+1}}dx + \int_1^\infty \frac{x}{\sqrt{x^4+1}}dx$$

then the convergence or divergence of the original integral is determined by the convergence or divergence of

$$\int_1^\infty \frac{x}{\sqrt{x^4+1}}dx$$

because the other component is finite. We can compare the integrand as follows:

$$\frac{x}{\sqrt{x^4+1}} > \frac{1}{2x}$$

Now we show this inequality to be true:

$$\begin{aligned} \frac{x}{\sqrt{x^4+1}} &> \frac{1}{2x} \\ \frac{x^2}{x^4+1} &> \frac{1}{4x^2} \\ 4x^4 &> x^4+1 \\ 3x^4 &> 1 \end{aligned}$$

This is true for all $x > 1$, which is why the rewriting of the integral above was done. Thus we see that

$$\int_1^\infty \frac{x}{\sqrt{x^4+1}}dx > \int_1^\infty \frac{1}{2x}dx = \frac{1}{2}\log(x)|_1^\infty$$

Thus by the comparison test

$$\int_0^\infty \frac{x}{\sqrt{x^4+1}}dx$$

diverges.

(∗∗) 10. Find the limit of the following sequence.

$$a_n = n - \sqrt{n+1}\sqrt{n+3}$$

Solution: We compute the limit as $n \to \infty$.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left(n - \sqrt{n+1}\sqrt{n+3}\right) \\
&= \lim_{n\to\infty} \frac{n - \sqrt{n+1}\sqrt{n+3}}{n + \sqrt{n+1}\sqrt{n+3}} \left(n + \sqrt{n+1}\sqrt{n+3}\right) \\
&= \lim_{n\to\infty} \frac{n^2 - (n+1)(n+3)}{n + \sqrt{n+1}\sqrt{n+3}} \\
&= \lim_{n\to\infty} \frac{n^2 - n^2 - 4n - 3}{n + \sqrt{n+1}\sqrt{n+3}} \\
&= \lim_{n\to\infty} \frac{-4n - 3}{n + \sqrt{n+1}\sqrt{n+3}} \\
&= \lim_{n\to\infty} \frac{-4 - 3/n}{1 + \sqrt{1+1/n}\sqrt{1+3/n}} \\
&= \frac{-4}{1+1} \\
&= -2
\end{aligned}$$

Thus the limit of the sequence as $n \to \infty$ is -2.

(∗∗) 11. For what real number(s) C does the following integral converge?

$$\int_0^\infty \left(\frac{x}{x^2+4} - \frac{C}{2x+3}\right)$$

Solution: We evaluate the integral:

$$\int_0^\infty \left(\frac{x}{x^2+4} - \frac{C}{2x+3}\right) = \lim_{k\to\infty} \left(\frac{1}{2}\log(x^2+4) - \frac{C}{2}\log(2x+3)\right)\Big|_0^k$$

We only need to care about the limit at ∞ since we are only investigating convergence.

$$\frac{1}{2}\lim_{k\to\infty} \left(\log(x^2+4) - C\log(2x+3)\right)\Big|_0^k = \frac{1}{2}\lim_{k\to\infty} \log\left(\frac{x^2+4}{(2x+3)^C}\right)$$

We want the ratio of polynomials in the logarithm to approach a (strictly) positive constant, which means that $C = 2$.

(∗∗) 12. Find the centroid of the region bounded by $9x^2 + 16y^2 = 144$ in the first quadrant.

Solution: First, solving for y, we have

$$y = \frac{1}{4}\sqrt{144 - 9x^2}$$

Then the area in the firrst quadrant will be given by

$$A = \int_0^4 \frac{1}{4}\sqrt{144 - 9x^2}dx = 3\pi$$

Note that the above integral can be easily computed by defining $z = 3x$, then using the area of a circle formula. Now we apply the formulas for the x and y coordinates of the centre of mass.

$$\begin{aligned}\overline{x} &= \frac{1}{A}\int_0^4 xf(x)dx \\ &= \int_0^4 x\frac{1}{4}\sqrt{144-9x^2}dx \\ &= \frac{16}{3\pi}\end{aligned}$$

Now for the y component,

$$\begin{aligned}\overline{y} &= \frac{1}{2A}\int_0^4 (f(x))^2dx \\ &= \frac{1}{2A}\int_0^4 \frac{144-9x^2}{16}dx \\ &= \frac{4}{\pi}\end{aligned}$$

Now the final answer is

$$(\overline{x}, \overline{y}) = \left(\frac{16}{3\pi}, \frac{4}{\pi}\right)$$

(∗∗) 13. A right circular cylindrical tank of radius 2 m and height 8 m is full of water. Find the work done in pumping the water to the top of the tank. Assume that the density of water is $1000\,\text{kg/m}^3$. You may assume that the gravitational field strength is $g = 10\,\text{m/s}^2$.

Solution: The volume of a thin horizontal slice is

$$dV = 2^2\pi dy = 4\pi dy$$

The mass of a thin horizontal slice is

$$dm = \rho dV = 4\rho\pi dy$$

The work to raise a thin horizontal slice by a distance y is

$$dW = gydm = 4\rho\pi gydy$$

Then the total work to raise all of the disks is given by

$$\begin{aligned}W &= \int_0^8 4\rho\pi gydy \\ &= 4\rho\pi g\int_0^8 ydy \\ &= 4\rho\pi g(32) \\ &= 128\rho\pi g \\ &= 1280000\pi\,\text{J}\end{aligned}$$

(∗∗) 14. Find an implicit solution to the following differential equation

$$\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}$$

Solution: First we separate variables:

$$\frac{1+2y^2}{y}dy = \cos x dx$$

Then we can integrate both sides with respect to the appropriate variable:

$$\begin{aligned} \int \frac{1+2y^2}{y}dy &= \int \cos x dx \\ \int \left(\frac{1}{y} + 2y\right) dy &= \sin x + C \\ \log|y| + y^2 &= \sin x + C \end{aligned}$$

Thus the final answer is

$$\log|y| + y^2 = \sin x + C$$

where we only have on constant of integration because one can be absorbed into another.

(∗∗) 15. A uniform 100 ft long cable weighing 5 lb/ft supports a safe weighing 500 lb. Find the work done in winding 80 ft of the cable onto a drum.

Solution: This problem is made easier if we consider the bottom 20 ft of cable as part of the safe, thus forming a $500 + (20)(5) = 600$ lb cable + safe. We will call the work to raise this cable + safe W_1. Then W_1 can be calculated as

$$W_1 = 600 \times 80 = 48000\,\text{ft}\cdot\text{lb}$$

Now consider the work W_2 it takes to wind 80 ft of cable onto a drum. The weight of a small length of cable is $5dy$, and the work it takes to raise this small length a distance of y is $dW_2 = 5ydy$. Then W_2 can be calculated as

$$W_2 = \int_0^{80} 5ydy = 16000$$

Now summing the two parts of wo rk together, we have

$$W_1 + W_2 = W_{\text{total}} = 64000\,\text{ft}\cdot\text{lb}$$

(∗∗) 16. Solve the following initial value problem explicitly in terms of y:

$$\frac{dy}{dx} = x^2y^2 + xy^2 + yx^2 + xy, \quad y(0) = 5$$

Solution: First we have to factor the RHS:

$$\frac{dy}{dx} = (x^2+x)(y^2+y)$$

Now we can separate variables:

$$\frac{dy}{y^2+y} = (x^2+x)dx$$

Since the variables are separated, we can integrate both sides with respect to the appropriate variable.

$$\begin{aligned} \int \frac{dy}{y^2+y} &= \int (x^2+x)dx \\ \int \left(\frac{1}{y} - \frac{1}{y+1}\right) dy &= \frac{x^3}{3} + x + C \\ \log|y| - \log|y+1| &= \frac{x^3}{3} + x + C \\ \log\left|\frac{y}{y+1}\right| &= \frac{x^3}{3} + x + C \\ \left|\frac{y}{y+1}\right| &= \tilde{C}e^{\frac{x^3}{3}+x} \end{aligned}$$

Now we plug in the initial condition to obtain that $\tilde{C} = 5/6$. Thus since both sides are positive we have

$$\frac{y}{y+1} = \frac{5}{6}e^{\frac{x^3}{3}+x}$$

Now solving for y, we have the final solution

$$y = \frac{(5/6)e^{\frac{x^3}{3}+x}}{1-(5/6)e^{\frac{x^3}{3}+x}}$$

(∗∗) 17. Determine the limit of the following sequence. Express your answer in terms of p.

$$\left\{\frac{1}{\sqrt[n]{n^p}}\right\}_{n=1}^{\infty}$$

Solution: For the sequence

$$\left\{\frac{1}{\sqrt[n]{n^p}}\right\}_{n=1}^{\infty}$$

Let the general term be denoted

$$f(n) = \frac{1}{\sqrt[n]{n^p}}$$

Since we will want to use L'Hopital's Rule, we change to a continuous variable x instead of a discrete variable n.

$$f(x) = x^{-p/x} = e^{\frac{-p\log x}{x}}$$

Now we need to take the limit $x \to \infty$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\frac{-p\log x}{x}}$$

This limit appears difficult to evaluate right away, so we instead examine the part of the limit inside the exponent:

$$\lim_{x\to\infty} \frac{-p\log x}{x} = \lim_{x\to\infty} \frac{\frac{-p}{x}}{1} = \lim_{x\to\infty} \frac{-p}{x} = 0$$

Therefore

$$\lim_{x\to\infty} e^{\frac{-p\log x}{x}} = e^0 = 1$$

So the sequence converges to 1 for any value of p.

$$\lim_{n\to\infty} f(n) = 1$$

(∗ ∗ ∗) 18. Determine if the following integral converges or diverges.

$$\int_0^1 \frac{1}{\sqrt{x}\log x}dx$$

Solution: First we note that for every $x \in (0,1)$,

$$\frac{1}{\sqrt{x}\log x} < 0$$

because the logarithm is always negative there. Looking at the integral, we see that at 0, $\log x$ goes to $-\infty$, and $\sqrt{x}$ goes to 0. So it is currently undetermined whether or not the integral converges there. However, at $x = 1$, $\log x$ goes to 0, but $\sqrt{x}$ goes to 1. This suggests that at $x = 1$, the integrand approaches $-\infty$, and thus the integral diverges to $-\infty$. Making the substitution $u = \log x$, $du = \frac{1}{x}dx = e^{-u}dx$, we have

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{x}\log x}dx &= \int_{-\infty}^0 \frac{e^u}{e^{u/2}u}du \\
&= \int_{-\infty}^0 \frac{e^{u/2}}{u}du \\
&= \int_{-\infty}^{-1} \frac{e^{u/2}}{u}du + \int_{-1}^0 \frac{e^{u/2}}{u}du
\end{aligned}$$

Looking only at the second integral

$$\int_{-1}^0 \frac{e^{u/2}}{u}du$$

we see that it diverges by the comparison test, since for $u \in (-1,0)$,

$$\frac{e^{u/2}}{u} < \frac{1}{\sqrt{e}u}$$

and the integral

$$\int_{-1}^0 \frac{1}{\sqrt{e}u}du$$

diverges to $-\infty$. Thus the original integral

$$\int_0^1 \frac{1}{\sqrt{x}\log x} dx$$

diverges to $-\infty$.

(∗∗∗) 19. (a) How much work is done in filling an upright cylindrical tank of radius 3 ft and height 10 ft with liquid weighing $w\,\mathrm{lb}/\mathrm{ft}^3$ through a hole in the bottom?

(b) How much if the tank is horizontal?

Solution:

(a) We can think about this problem as lifting slices of water of circular cross section from the bottom of the tank up to some height y inside the tank. The volume of a thin circular slice is

$$dV = \pi 3^2 dy = 9\pi dy$$

The weight of a thin circular slice is

$$d(\text{weight}) = wdV = 9\pi wdy$$

The work to raise a thin circular slice by a distance y is

$$dW = yd(\text{weight}) = 9\pi wydy$$

Thus we can calculate the work by integrating over all slices from $y = 0$ ft to $y = 10$ ft.

$$W = \int_0^{10} 9\pi wydy = 450\pi \text{ ft} \cdot \text{lb}$$

(b) We take the same approach as above, however each slice of water that is lifted has a different cross sectional area. It is no longer uniformly circular cross sections like in part (a). Consider rectangular cross sections with dimensions 10 and $2x$. The volume of each cross section will be

$$dV = Ady = (10)(2x)dy = 20xdy$$

Then we can calculate the weight of a thin rectangular slice as

$$d(\text{weight}) = wdV = 20wxdy$$

The work to raise a thin rectangular sice a distance y is

$$dW = 20xywdy$$

Since we are dealing with cross sections of a cylinder, x and y are related by $x^2 + (y-3)^2 = 3^2$, because y is measured from the very bottom of the cylinder, not from the centre. Thus we can substitute x in terms of y in order to evaluate the integral.

$$W = 20w \int_0^6 y\sqrt{9-(y-3)^2}dy$$

To evaluate this integral, let $y - 3 = u$. Then we have

$$\begin{aligned} W &= 20w \int_{-3}^{3} (u+3)\sqrt{9-u^2}du \\ &= 20w \int_{-3}^{3} u\sqrt{9-u^2}du + 60w \int_{-3}^{3} \sqrt{9-u^2}du \end{aligned}$$

The first integral is zero because it is an odd function evaluated over a symmetric interval, and the second one can be evaluated geometrically because it is the integral of the graph of a circle. Then we have

$$W = 270w\pi \,\text{ft} \cdot \text{lb}$$

(∗∗∗) 20. Find the area between the curve $y^2 = \dfrac{x^2}{1-x^2}$ and its vertical asymptotes.

Solution: First we need to determine what the graph looks like.

- The graph is even with respect to both the x and y axes, because replacing x by $-x$ leaves the equation unchanged, and replacing y by $-y$ also leaves the equation unchanged.
- There will be vertical asymptotes at $x = 1$ and $x = -1$.

Thus we can find the area enclosed in the first quadrant between 0 and 1, then multiply that result by 4. Let A be the total area. Thus in the first quadrant,

$$y = \frac{x}{\sqrt{1-x^2}}$$

Now we integrate.

$$\begin{aligned} \frac{A}{4} &= \int_0^1 \frac{x}{\sqrt{1-x^2}}dx \\ &= \lim_{k\to 1^-} \int_0^k \frac{x}{\sqrt{1-x^2}}dx \\ &= -\lim_{k\to 1^-} (1-x^2)^{1/2}\Big|_0^k \\ &= -(0-1) \\ &= 1 \end{aligned}$$

Thus the total area is $A = 4$