Mathematics 101 Quiz 5 Review Package – **Solutions**

UBC Engineering Undergraduate Society

Attempt questions to the best of your ability. This review package consists of 13 pages, including 1 cover page and 23 questions. The questions are meant to be the level of a real examination or slightly above, in order to prepare you for the real exam. Material from lectures and from the relevant textbook sections is examinable, and the problems for this package were chosen with that in mind, as well as considerations based on past examination question difficulty and style. Problems are ranked in difficulty as $(*)$ for easy, $(**)$ for medium, and $(***)$ for difficult. Note that sometimes difficulty can be subjective, so do not be discouraged if you are stuck on a $(*)$ problem.

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Some of the problems in this package were not created by the EUS. Those problems originated from one of the following sources:

- Schuam's Outline of Calculus 2 ed; Ayres Jr., Frank
- Calculus – Early Transcendentals 7 ed; Stewart, James
- Calculus – 3 ed; Spivak, Michael
- Calculus Volume 1 2 ed; Apostol, Tom

All solutions prepared by the EUS.

Good Luck!

(∗∗) 1. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2-1}}$$

Solution: Apply the limit comparison test: Compare with $\sum n^{-2/3}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^{-2/3}}{\frac{1}{\sqrt[3]{n^2-1}}} &= \lim_{n\to\infty} \frac{\sqrt[3]{n^2-1}}{n^{2/3}} \\
&= \lim_{n\to\infty} \sqrt[3]{\frac{n^2-1}{n^2}} \\
&= \lim_{n\to\infty} \sqrt[3]{1-\frac{1}{n^2}} \\
&= \sqrt[3]{1} \\
&= 1 \\
&< \infty
\end{aligned}$$

Since this limit equals $1 < \infty$, and $\sum n^{-2/3}$ diverges by the p-test $(2/3 < 1)$, the given series also diverges.

(∗) 2. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \sin\left(\frac{2n+1}{3-2n}\right)$$

Solution: This series diverges by the divergence test. Taking the limit of the term

$$a_n = \sin\left(\frac{2n+1}{3-2n}\right)$$

as n goes to infinity:

$$\lim_{n\to\infty} \sin\left(\frac{2n+1}{3-2n}\right) = \sin(-1) \neq 0$$

Thus the limit of the general term does not approach zero, so the series diverges.

(∗) 3. Determine if the following series converges or diverges.

$$\sum_{n=3}^{\infty} \sqrt{\frac{3e^n+1}{e^n-1}}$$

Solution: This series diverges by the divergence test. Taking the limit of hte term

$$a_n = \sqrt{\frac{e^{2n}+1}{e^n-1}}$$

as n goes to infinity:

$$\lim_{n\to\infty} \sqrt{\frac{3e^n+1}{e^n-1}} = \sqrt{3} \neq 0$$

Thus the limit of the general term does not approach zero, so the series diverges.

(∗) 4. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$$

Solution: Apply the limit comparison test. Compare with $\sum 1/n$.

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{n^2}{n^3+1}} = \lim_{n\to\infty} \frac{n^3+1}{n^3} = \lim_{n\to\infty} 1 + \frac{1}{n^3} = 1 < \infty$$

Thus, since $\sum 1/n$ diverges by the p-test, the given series also diverges.

(∗) 5. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\log n}{n}$$

Solution: Apply the comparison test. For all $n \geq 3$ we have,

$$\log n > 1$$

Thus we can divide by n to obtain

$$\frac{\log n}{n} > \frac{1}{n}$$

Since the series $\sum \frac{1}{n}$ diverges by the p-test, the given series also diverges.

(∗) 6. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{\log n}$$

Solution: Apply the comparison test. For $n \geq 2$ we have,

$$n < e^n$$

taking the logarithm of both sides

$$\log n < n$$

This means that

$$\frac{1}{\log n} > \frac{1}{n}$$

Since the series $\sum \frac{1}{n}$ diverges by the p-test, the series

$$\sum_{n=2}^{\infty} \frac{1}{\log n}$$

also diverges.

(∗) 7. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$$

Solution: Apply the alternating series test:

- Let $a_n = \log n/n = f(n)$. Consider the continuous function $f(x) = \log x/x$. The derivative of the function f is $f'(x) = (1 - \log x)/x^2$, which is negative for all $x > e$. Thus $f(x)$ is decreasing for all $x > e$, which means that a_n is also decreasing for all $n \geq 3$.
- Now compute the limit $\lim_{n\to\infty} \log n/n = \lim_{x\to\infty} \log x/x \stackrel{L'H}{=} \lim_{x\to\infty} 1/x = 0$.

Thus, the series converges because the limit of the terms go to zero, and the absolute value of the terms decreases.

(∗) 8. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$$

Solution: Apply the limit comparison test: Compare with

$$\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$$

Taking the limit of the ratio of the two general terms of the respective series:

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^{-2/3}}{\frac{1}{\sqrt[3]{n^2+1}}} &= \lim_{n\to\infty} \frac{\sqrt[3]{n^2+1}}{n^{2/3}} \\
&= \lim_{n\to\infty} \sqrt[3]{\frac{n^2+1}{n^2}} \\
&= \lim_{n\to\infty} \sqrt[3]{1+\frac{1}{n^2}} \\
&= \sqrt[3]{1} \\
&= 1 \\
&< \infty
\end{aligned}$$

Since this limit equals $1 < \infty$, and $\sum n^{-2/3}$ diverges by the p-test ($2/3 < 1$), the given series also diverges.

(∗∗) 9. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}$$

Solution: Apply the integral test. Consider

$$\int_2^{\infty} \frac{1}{x \log x} dx = \lim_{k\to\infty} \log(\log x)\Big|_2^k$$

This integral diverges, so the series also diverges.

(∗∗) 10. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$$

Solution: Apply the integral test. Consider

$$\int_2^{\infty} \frac{1}{x(\log x)^2} dx = \lim_{k\to\infty} \left.\frac{-1}{\log x}\right|_2^k = \frac{1}{\log 2}$$

Since the integral converges, the given series also converges.

(∗∗) 11. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$$

Solution: Apply the integral test. Consider

$$\int_2^\infty \frac{1}{x^2 \log x} dx$$

and let $u = \log x$, $du = dx/x$ Thus we have

$$\int_{\log 2}^\infty \frac{e^{-u}}{u} du < \int_{\log 2}^\infty e^{-u} du = \frac{1}{2} < \infty$$

Thus the integral converges, so the series also converges.

(∗∗) 12. Solve the following equation for c:

$$\sum_{n=2}^\infty (1+c)^{-n} = 2$$

Solution: Let $1/(1+c) = r$. Under the assumption that $r = 1/|1+c| < 1$, we can sum this series. First we must rewrite it as follows:

$$\begin{aligned}
\sum_{n=2}^\infty r^n &= -1 - r + \sum_{n=0}^\infty r^n \\
&= -1 - r + \frac{1}{1-r} \\
&= -1 - \frac{1}{1+c} + \frac{1}{1 - \frac{1}{1+c}} \\
&= -1 - \frac{1}{1+c} + \frac{1+c}{1+c-1} \\
&= -1 - \frac{1}{1+c} + \frac{1+c}{1+c-1} \\
&= -1 - \frac{1}{1+c} + \frac{1+c}{c} \\
&= -1 - \frac{1}{1+c} + 1 + \frac{1}{c} \\
&= \frac{1}{c} - \frac{1}{1+c} \\
&= \frac{1}{c(c+1)} \\
&= 2
\end{aligned}$$

Now solving the equation

$$2c^2 + 2c - 1 = 0$$

for c using the quadratic formula,

$$c = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-1 \pm \sqrt{3}}{2}$$

Note if we take $c = \frac{-1-\sqrt{3}}{2}$, then $1 + c = \frac{1-\sqrt{3}}{2}$, but $1/|1+c| > 1$ in this case. Thus we must reject it, and take

$$c = \frac{-1+\sqrt{3}}{2}$$

(∗∗) 13. Express the number $0.\overline{1234}$ as a ratio of two integers. You do not need to fully simplify your answer.

Solution: The number $0.\overline{1234}$ can be written as

$$0.123412341234\dots$$

which suggests the following form:

$$0.1234\left(1+\frac{1}{10^4}+\frac{1}{10^8}+\cdots\right)$$

which we recognize as a geometric series

$$\frac{1234}{10000}\sum_{n=0}^{\infty}(10^{-4})^n$$

The common ratio is clearly less than 1, so we can sum to form

$$\begin{aligned}\frac{1234}{10000}\left(\frac{1}{1-\frac{1}{10000}}\right) &= \frac{1234}{10000}\frac{10000}{10000-1}\\ &= \frac{1234}{9999}\end{aligned}$$

(∗∗) 14. Express the number $1.35\overline{42}$ as a ratio of two integers. You do not need to fully simplify your answer.

Solution: The number $1.37\overline{42}$ can be written as

$$1.37424242\cdots = 1.37 + 0.00424242\dots$$

which suggests the following form,

$$\begin{aligned}1.37+0.0042\left(1+\frac{1}{10^2}+\frac{1}{10^4}+\cdots\right) &= \frac{137}{100}+\frac{42}{10000}\sum_{n=0}^{\infty}\left(10^{-2}\right)^n\\ &= \frac{137}{100}+\frac{42}{10000}\left(\frac{1}{1-\frac{1}{100}}\right)\\ &= \frac{137}{100}+\frac{42}{10000}\left(\frac{100}{100-1}\right)\\ &= \frac{137}{100}+\frac{42}{100}\left(\frac{1}{99}\right)\\ &= \frac{137\times 99+42}{100\times 99}\\ &= \frac{13700-137+42}{9900}\\ &= \frac{13605}{9900}\\ &= \frac{907}{660}\end{aligned}$$

(∗∗) 15. For what real number(s) C does the following series converge?

$$\sum_{n=0}^{\infty}\left(\frac{n}{n^2+1}-\frac{C}{5n+1}\right)$$

Solution: We can apply the integral test in this case. The given series converges if and only if the integral

$$\int_0^{\infty}\left(\frac{x}{x^2+1}-\frac{C}{5x+1}\right)dx$$

converges. Now we evaluate the integral:

$$\begin{aligned}\int_0^{\infty}\left(\frac{x}{x^2+1}-\frac{C}{5x+1}\right)dx &= \lim_{k\to\infty}\int_0^{k}\left(\frac{x}{x^2+1}-\frac{C}{5x+1}\right)dx \\ &= \lim_{k\to\infty}\left(\frac{1}{2}\log(x^2+1)-\frac{C}{5}\log(5x+1)\right)\Big|_0^k \\ &= \lim_{k\to\infty}\log\left(\frac{(k^2+1)^{1/2}}{(5k+1)^{C/5}}\right)\end{aligned}$$

We want the exponents on the k's in the numerator and denominator to be the same as k grows large, so we set $1 = C/5$, which yields $C = 5$.

(∗∗) 16. Find the limit of the following sequence:

$$\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\right\}$$

Solution: This sequence can be rewritten as

$$\left\{2^{1/2}, \left(2^{1+1/2}\right)^{1/2}, \left(2^{1+1/2+1/4}\right)^{1/2}, \ldots\right\}$$

which can be further simplified to

$$\left\{2^{1/2}, 2^{1/2+1/4}, 2^{1/2+1/4+1/8}, \ldots\right\}$$

Thus we see a pattern in the terms. The pattern is

$$a_n = 2^{\sum_{k=1}^{n}\frac{1}{2^k}}$$

Taking the limit, we have

$$\lim_{n\to\infty} a_n = 2^{\sum_{k=1}^{\infty}\frac{1}{2^k}}$$

The series in the exponent can be evaluated to

$$\lim_{n\to\infty} a_n = 2^1 = 2$$

Thus the sequence converges to 2.

(∗∗) 17. Determine if the following series converges or diverges. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Solution: Since $n \in \mathbb{N}$, we know that $0 < 1/n < 1$. Thus, $\sin(1/n) > 0$. We can then use the limit comparison test, comparing with $\sum 1/n$

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\sin(1/n)}$$

Let $1/n = x$, so then we have

$$\lim_{x\to 0} \frac{x}{\sin x} \overset{L'H}{=} \lim_{x\to 0} \frac{1}{\cos x} = 1 < \infty$$

Thus since $\sum 1/n$ diverges, the given series also diverges.

(∗∗) 18. Determine how many terms of the series we need to sum before the difference between the partial sum and the sum of the series is $|\text{error}| < 0.0001$.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\log(n+1)}$$

Solution: We apply the alternating series estimation theorem. We set

$$b_{n+1} = \frac{1}{\log(n+2)}$$

and then we can solve the following inequality for n:

$$\begin{aligned} b_{n+1} &< 0.0001 \\ \frac{1}{\log(n+2)} &< 10^{-4} \\ 10^4 &< \log(n+2) \\ e^{10^4} - 2 < n & \end{aligned}$$

Thus if

$$n > e^{10^4} - 2$$

(we could also round the value $e^{10^4} - 2$ up to the nearest integer) the partial sum

$$S_{e^{10^4}-1} = \sum_{n=1}^{e^{10^4}} \frac{(-1)^n}{\log(n+1)}$$

will be within 0.0001 of the actual sum of the series.

(∗∗) 19. Determine how many terms of the series need to be summed in order for the partial sum to be correct to 3 decimal places.

$$\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n}$$

Solution: This series can be rewritten as

$$\sum_{n=1}^{\infty} \frac{n(-1)^n}{2^n}$$

and we define

$$b_{n+1} = \frac{n+1}{2^{n+1}}$$

and then applying the alternating series estimation theorem, since we need the partial sum to be correct to three decimal places, the error should be less than 10^{-3}.

$$\begin{aligned} \frac{n+1}{2^{n+1}} &< 10^{-3} \\ 1000(n+1) &< 2^{n+1} \end{aligned}$$

Since $2^{10} = 1024$, we know that $n > 9$. Trying other values of n gives us that

$$14000 = 1000 \cdot 14 < 2^{14} = 16284$$

Thus $n = 13$ suffices to bring our partial sum correct to 3 decimal places of the sum of the series.

($* * *$) 20. Determine for which (if any) natural numbers k the following series converges.

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^k}$$

Solution: Apply the integral test. Consider

$$\int_2^{\infty} \frac{1}{(\log x)^k} dx$$

Make the substitution $\log x = u$, $du = \frac{dx}{x}$, $dx = e^u du$. Thus we have

$$\int_2^{\infty} \frac{1}{(\log x)^k} dx = \int_{\log 2}^{\infty} \frac{e^u}{u^k} du$$

This integral diverges if the integrand has a nonzero limit as $u \to \infty$. Compute the limit.

$$\lim_{u\to\infty} \frac{e^u}{u^k} \overset{L'H}{=} \lim_{u\to\infty} \frac{e^u}{ku^{k-1}}$$

Performing L'Hopital's rule $k-1$ more times, we have that the limit diverges to ∞. Thus the integral also diverges, and the given series diverges too, for all natural numbers k.

($* * *$) 21. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$$

Solution: If $n \geq 3$, it is clear that $n > 2\log n$. And since $\log\log n > 0$ for $n \geq 3$, we have the inequality

$$n \log\log n > 2\log n$$

Thus $\log(\log n)^n > \log n^2$, and $(\log n)^n > n^2$. This implies that

$$0 < \frac{1}{(\log n)^2} < n^2$$

Since $\sum 1/n^2$ converges, this means that the given series also converges.

(∗ ∗ ∗) 22. Find the sum of the series. Hint: Expand in terms of partial fractions.

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

Solution: First we split the expression

$$\frac{1}{n^2-1} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

in terms of partial fractions as shown. Then define the partial sum

$$S_N = \sum_{n=2}^{N} \frac{1}{n^2-1}$$

We can then evaluate S_N:

$$\begin{aligned}
S_N &= \sum_{n=2}^{N} \frac{1}{n^2-1} \\
&= \frac{1}{2}\sum_{n=2}^{N}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) \\
&= \frac{1}{2}\left(\sum_{n=2}^{N}\frac{1}{n-1} - \sum_{n=2}^{N}\frac{1}{n+1}\right) \\
&= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \sum_{n=4}^{N}\frac{1}{n-1} - \sum_{n=4}^{N+2}\frac{1}{n-1}\right) \\
&= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \sum_{n=4}^{N}\frac{1}{n-1} - \left(\sum_{n=4}^{N}\frac{1}{n-1} + \frac{1}{N} + \frac{1}{N+1}\right)\right) \\
&= \frac{1}{2}\left(\frac{3}{2} - \frac{1}{N} - \frac{1}{N+1}\right)
\end{aligned}$$

Now taking the limit

$$\lim_{N\to\infty} S_N = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{N} - \frac{1}{N+1}\right) = \frac{3}{4}$$

Thus the sum is

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \frac{3}{4}$$

(∗ ∗ ∗) 23. Consider the Fibonacci sequence defined by

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}$$

(a) Prove that $\dfrac{1}{f_{n-1}f_{n+1}} = \dfrac{1}{f_{n-1}f_n} - \dfrac{1}{f_n f_{n+1}}$.

(b) Using the identity proved in part (a), find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{f_{n-1}f_{n+1}}$.

(c) Again using the identity proved in part (a), find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}}$.

Solution:

(a) We can use the formula for the $n+1$ term to derive the given identity. Since $f_n = f_{n-1} + f_{n-2}$, we also know that $f_{n+1} = f_n + f_{n-1}$, so $f_{n+1} - f_{n-1} = f_n$. Starting with the the right hand side

$$\begin{aligned}
\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}} &= \frac{1}{f_n}\left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right) \\
&= \frac{1}{f_n}\left(\frac{f_{n+1} - f_{n-1}}{f_{n+1}f_{n-1}}\right) \\
&= \frac{1}{f_n}\left(\frac{f_n}{f_{n+1}f_{n-1}}\right) \\
&= \frac{1}{f_{n-1}f_{n+1}}
\end{aligned}$$

(b) This is an exercise in telescoping series. Using the identity proved in part (a), we can rewrite the given series as a telescoping series.

$$\sum_{n=2}^{\infty} \frac{1}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty} \left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}}\right)$$

Now writing this in terms of partial sums so that we can later take the limit, we have

$$\begin{aligned} S_N &= \sum_{n=2}^{N}\left(\frac{1}{f_{n-1}f_n}-\frac{1}{f_nf_{n+1}}\right) \\ &= \sum_{n=2}^{N}\frac{1}{f_{n-1}f_n}-\sum_{n=2}^{N}\frac{1}{f_nf_{n+1}} \\ &= \frac{1}{f_1f_2}+\sum_{n=3}^{N}\frac{1}{f_{n-1}f_n}-\left(\sum_{n=3}^{N+1}\frac{1}{f_{n-1}f_n}\right) \\ &= \frac{1}{f_1f_2}+\sum_{n=3}^{N}\frac{1}{f_{n-1}f_n}-\left(\sum_{n=3}^{N}\frac{1}{f_{n-1}f_n}+\frac{1}{f_Nf_{N+1}}\right) \\ &= \frac{1}{f_1f_2}-\frac{1}{f_Nf_{N+1}} \end{aligned}$$

Now taking the limit

$$\lim_{N\to\infty}S_N=\frac{1}{f_1f_2}=1$$

Thus the series is

$$\sum_{n=2}^{\infty}\frac{1}{f_{n-1}f_{n+1}}=1$$

(c) Multiplying both sides of the identity proved in part (a) by f_n, we have

$$\frac{f_n}{f_{n+1}f_{n-1}}=\frac{1}{f_{n-1}}-\frac{1}{f_{n+1}}$$

Then we can rewrite the series as

$$\sum_{n=2}^{\infty}\frac{f_n}{f_{n-1}f_{n+1}}=\sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}}-\frac{1}{f_{n+1}}\right)$$

Now writing this in terms of partial sums so that we can later take the limit, we have

$$\begin{aligned} S_N &= \sum_{n=2}^{N}\left(\frac{1}{f_{n-1}}-\frac{1}{f_{n+1}}\right) \\ &= \sum_{n=2}^{N}\frac{1}{f_{n-1}}-\sum_{n=2}^{N}\frac{1}{f_{n+1}} \\ &= \frac{1}{f_1}+\frac{1}{f_2}+\sum_{n=4}^{N}\frac{1}{f_{n-1}}-\left(\sum_{n=4}^{N+2}\frac{1}{f_{n-1}}\right) \\ &= \frac{1}{f_1}+\frac{1}{f_2}+\sum_{n=4}^{N}\frac{1}{f_{n-1}}-\left(\sum_{n=4}^{N}\frac{1}{f_{n-1}}+\frac{1}{f_N}+\frac{1}{f_{N+1}}\right) \\ &= \frac{1}{f_1}+\frac{1}{f_2}-\frac{1}{f_N}-\frac{1}{f_{N+1}} \end{aligned}$$

Now taking the limit

$$\lim_{N\to\infty}S_N=\frac{1}{f_1}+\frac{1}{f_2}=2$$