Mathematics 152 Midterm 2 Review Package –

UBC Engineering Undergraduate Society

Problems are ranked in difficulty as ($*$) for easy, ($**$) for medium, and ($* * *$) for difficult. Note that sometimes difficulty can be subjective, so do not be discouraged if you are stuck on a ($*$) problem.

Solutions posted at: `https://ubcengineers.ca/tutoring`

If you believe that there is an error in these solutions, or have any questions, comments, or suggestions regarding EUS Tutoring sessions, please e-mail us at: tutoring@ubcengineers.ca. If you are interested in helping with EUS tutoring sessions in the future or other academic events run by the EUS, please e-mail vpacademic@ubcengineers.ca.

Some of the problems in this package were not created by the EUS. Those problems originated from one of the following sources:

- Schuam's Outline of Matrix Operations; Richard Bronson
- Calculus 7th ed; James Stewart
- Linear Algebra; Sterling K. Berberian
- Linear Algebra and Its Applications 3rd ed; Gilbert Strang
- Linear Algebra and Matrix Theory; Robert Stoll

All solutions prepared by the EUS.

EUS Health and Wellness Study Tips

- **Eat Healthy**—Your body needs fuel to get through all of your long hours studying. You should eat a variety of food (not just a variety of ramen) and get all of your food groups in.
- **Take Breaks**—Your brain needs a chance to rest: take a fifteen minute study break every couple of hours. Staring at the same physics problem until your eyes go numb wont help you understand the material.
- **Sleep**—We have all been told we need 8 hours of sleep a night, university should not change this. Get to know how much sleep you need and set up a regular sleep schedule.

Good Luck!

(∗) 1. Consider the linear system

$$\begin{cases} x + 2y + z = 1 \\ -x + 3z = 1 \\ x - y - 3z = 0 \end{cases}$$

(a) Write this system as an augmented matrix.

(b) Write the system to row echelon form

(c) Write the system in reduced row echelon form

(d) Find the solution to the system

Solution:

(a) Taking the coefficients and putting them in the matrix, and constants on the right of the line,

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 0 & 3 & 1 \\ 1 & -1 & -3 & 0 \end{array}\right)$$

(b) Performing elimination until all of the elements below the diagonal, we obtain

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

(c) Continuing elimination until there are only ones on the diagonal, we obtain

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

(d) Thus, $x = 2$, $y = -1$, $z = 1$.

(∗) 2. Compute the rank of $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 4 & 0 & 2 \end{pmatrix}$

Solution: Performing elimination, we obtain

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 4 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and see that there are now only two independent rows. Note that there are also only two independent columns. Thus $\text{rank}(A) = 2$

(∗) 3. (a) Find the work done in moving an object along a vector $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ if the applied force is $\mathbf{F} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

(b) Find the angle between the applied force and the displacement.

Solution:

(a) $W = \mathbf{r} \cdot \mathbf{F} = 6 - 2 + 5 = 9J$

(b) We can find the angle using the dot product: $\mathbf{r} \cdot \mathbf{F} = 9 = |\mathbf{r}| \cdot |\mathbf{F}| \cos\theta$. Dividing through by $|\mathbf{r}|$ and $|\mathbf{F}|$, then inverting the cosine, we have

$$\theta = \arccos\left(\frac{9}{\sqrt{6} \cdot \sqrt{38}}\right)$$

(**) 4. Consider the following lines of Matlab code:

```
x = 1:7;
y = 1:0.3:1.7;
```

(a) What is `x`?

(b) What is `y`?

(c) If you call `sin(y)`, what will the output be? If this operation is defined, you may leave your answers in terms of trigonometric functions.

(d) Is `cross(x,y)` defined?

Solution:

(a) The notation means the vector starting with 1, then incrementing by 1 until 7. That is,

```
x = [1 2 3 4 5 6 7]
```

(b) The notation means the vector starting with 1, then incrementing by 0.3 until 1.7. Note that it will not surpass 1.7

```
y = [1 1.3 1.6]
```

(c) Calling `sin(y)` will apply the sin function to every entry in the vector. That is,

```
sin(y) = [sin(1) sin(1.3) sin(1.6)]
```

(d) The operation `cross(x,y)` is not defined because only `y` has three components. Since `x` has 7 components, it cannot enter into the cross product.

(*) 5. What matrix $A : \mathbb{R}^2 \to \mathbb{R}^2$ represents projection onto the x axis followed by projection onto the y axis?

Solution: The zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ performs this operation, because every vector will be mapped into the zero vector. Projecting a vector onto the x - axis will remove its y - component, and when projecting onto the y - axis, there is already no y - component when the x component is removed the zero vector is produced.

(*) 6. If $A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 & 1 \\ 2 & -2 & -2 \\ -1 & 2 & 1 \end{pmatrix}$ $C = \begin{pmatrix} 3 & 1 & -3 \\ 0 & 2 & 6 \\ -1 & 2 & 1 \end{pmatrix}$

Compute

(a) AB

(b) AC

What can you say about AB and AC? What does it say about cancellation of matrices? Does $AB = AC$ imply that $B = C$?

Solution:

(a)

$$\begin{aligned} AB &= \begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 2 & -2 & -2 \\ -1 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4(2)+2(2)+0(-1) & 4(3)+2(-2)+0(2) & 4(1)+2(-2)+0(1) \\ 2(2)+1(2)+0(-1) & 2(3)+1(-2)+0(2) & 2(1)+1(-2)+0(1) \\ -2(2)+(-1)(2)+1(-1) & -2(3)+(-1)(-2)+1(2) & -2(1)+(-1)(-2)+1(1) \end{pmatrix} \\ &= \begin{pmatrix} 12 & 8 & 0 \\ 6 & 4 & 0 \\ -7 & -2 & 1 \end{pmatrix} \end{aligned}$$

(b)

$$\begin{aligned} AC &= \begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & -3 \\ 0 & 2 & 6 \\ -1 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4(3)+2(0)+0(-1) & 4(1)+2(2)+0(2) & 4(-3)+2(6)+0(1) \\ 2(3)+1(0)+0(-1) & 2(1)+1(2)+0(2) & 2(-3)+1(6)+0(1) \\ -2(3)+(-1)(0)+1(-1) & -2(1)+(-1)(2)+1(2) & -2(-3)+(-1)(6)+1(1) \end{pmatrix} \\ &= \begin{pmatrix} 12 & 8 & 0 \\ 6 & 4 & 0 \\ -7 & -2 & 1 \end{pmatrix} \end{aligned}$$

Remark 1. *$AC = AB$, but $B \neq C$. This shows that the cancellation law is not always valid for matrices. The cancellation law in this case is not valid because* $\det A = 0$*. This means that one could* ***not*** *multiply both sides of $AC = AB$ by A^{-1} to obtain $B = C$.*

(∗) 7. Compute the determinant of the matrix: $\begin{pmatrix} 2 & 6 & \log 2 & \pi^2 & e \\ 0 & 5 & 2 & 4 & \sqrt{5} \\ 0 & 0 & \pi & \sin(9) & 7 \\ 0 & 0 & 0 & -4 & 21 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$

Solution: The determinant of a triangular matrix is just the product of its diagonal entries.

$$\begin{vmatrix} 2 & 6 & \log 2 & \pi^2 & e \\ 0 & 5 & 2 & 4 & \sqrt{5} \\ 0 & 0 & \pi & \sin(9) & 7 \\ 0 & 0 & 0 & -4 & 21 \\ 0 & 0 & 0 & 0 & 6 \end{vmatrix} = (2)(5)(\pi)(-4)(6) = -240\pi$$

(∗) 8. What matrix $A : \mathbb{R}^2 \to \mathbb{R}^2$ represents projection onto the x axis followed by projection onto the y axis?

Solution: The zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ performs this operation, because every vector will be mapped into the zero vector. Projecting a vector onto the x - axis will remove its y - component, and when projecting onto the y - axis, there is already no y - component when the x component is removed the zero vector is produced.

(∗) 9. Compute the transpose of $A = \begin{pmatrix} -6 & 9 & 0 \\ 1 & -1 & 0 \\ 2 & \pi & 3 \\ 5 & 2 & 6 \end{pmatrix}$

Solution: To calculate the transpose of a matrix, simply swap the rows and columns (e.g. column 1 values of A going down become row 1 values going right).
With

$$A = \begin{pmatrix} -6 & 9 & 0 \\ 1 & -1 & 0 \\ 2 & \pi & 3 \\ 5 & 2 & 6 \end{pmatrix}$$

We compute

$$A^T = \begin{pmatrix} -6 & 1 & 2 & 5 \\ 9 & -1 & \pi & 2 \\ 0 & 0 & 3 & 6 \end{pmatrix}$$

(∗) 10. (a) Compute the product $A\mathbf{x} = \begin{pmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

(b) Without computing the determinant, determine if the matrix A is invertible or not.

Solution:

(a) $A\mathbf{x} = \begin{pmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3(2) + (-6)(1) + 0(1) \\ 0(2) + 2(1) + (-2)(1) \\ 1(2) + (-1)(1) + (-1)(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Any multiple of $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ would also produce the zero vector.

(b) The matrix is not invertible because it maps nonzero vectors to the zero vector.

(∗) 11. Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 8 \\ 0 & -9 \end{pmatrix}$,

(a) If it is defined, compute AB

(b) If it is defined, compute BA

Solution:

(a) Since the matrix dimensions don't match for multiplication,

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{pmatrix} \begin{pmatrix} 7 & 8 \\ 0 & -9 \end{pmatrix} = \text{undefined}$$

(b) Since the matrix dimensions match for multiplication,

$$\begin{aligned} BA &= \begin{pmatrix} 7 & 8 \\ 0 & -9 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 7(1) + 8(4) & 7(2) + 8(-5) & 7(3) + 8(6) \\ 0(1) + (-9)(4) & 0(2) + (-9)(-5) & 0(3) + (-9)(6) \end{pmatrix} \\ &= \begin{pmatrix} 39 & -26 & 69 \\ -36 & 45 & -54 \end{pmatrix} \end{aligned}$$

(∗) 12. Given $T(\mathbf{x}) = \begin{pmatrix} -1 & 3 \\ 9 & 4 \end{pmatrix}\mathbf{x}$, and $S(\mathbf{x}) = \begin{pmatrix} 3 & -2 & 6 \\ -4 & 6 & 2 \end{pmatrix}\mathbf{x}$, compute the following (if defined)

(a) $T \circ S$

(b) $S \circ T$

(c) $T\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$

(d) $S\left(\begin{pmatrix} -2 \\ 4 \end{pmatrix}\right)$

(e) $S\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right)$

Solution:

(a)

$$\begin{aligned} T \circ S &= \begin{pmatrix} -1 & 3 \\ 9 & 4 \end{pmatrix}\begin{pmatrix} 3 & -2 & 6 \\ -4 & 6 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -1(3)+3(-4) & -1(-2)+3(6) & -1(6)+3(2) \\ 9(3)+4(-4) & 9(-2)+4(6) & 9(6)+4(2) \end{pmatrix} \\ &= \begin{pmatrix} -15 & 20 & 0 \\ 11 & 6 & 62 \end{pmatrix} \end{aligned}$$

(b) $S \circ T = \text{undefined}$

(c)

$$\begin{aligned} T\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) &= \begin{pmatrix} -1 & 3 \\ 9 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1(2)+3(1) \\ 9(2)+4(1) \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 22 \end{pmatrix} \end{aligned}$$

(d) $S\left(\begin{pmatrix} -2 \\ 4 \end{pmatrix}\right) = \text{undefined}$

(e)

$$\begin{aligned} S\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) &= \begin{pmatrix} 3 & -2 & 6 \\ -4 & 6 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 3(1)+(-2)(2)+6(3) \\ -4(1)+6(2)+2(3) \end{pmatrix} \\ &= \begin{pmatrix} 17 \\ 14 \end{pmatrix} \end{aligned}$$

(∗) 13. (a) Find the matrix $R : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates vectors by 225°counterclockwise

(b) Find the image of $(2, 5)$ under this linear transformation.

Solution:

(a) The general rotation matrix is given by

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and we need to evaluate this rotation matrix for $\theta = 5\pi/4$. Thus

$$\begin{aligned} R &= \begin{pmatrix} \cos(5\pi/4) & -\sin(5\pi/4) \\ \sin(5\pi/4) & \cos(5\pi/4) \end{pmatrix} \\ &= \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \\ &= \frac{-1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

(b) Applying this matrix to the given vector:

$$\begin{aligned} R\left(\begin{pmatrix} 2 \\ 5 \end{pmatrix}\right) &= \frac{-1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ &= \frac{-1}{\sqrt{2}} \begin{pmatrix} 1(2) + (-1)(5) \\ 1(2) + 1(5) \end{pmatrix} \\ &= \frac{-1}{\sqrt{2}} \begin{pmatrix} -3 \\ 7 \end{pmatrix} \end{aligned}$$

(∗∗) 14. What matrix has the effect of rotating a vector $v \in \mathbb{R}^2$ through 90°clockwise, and then projecting the result onto the x axis?

Solution: The rotation matrix will be

$$R = \begin{pmatrix} \cos(-90°) & -\sin(-90°) \\ \sin(-90°) & \cos(-90°) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the projection matrix will be

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Multiplying the two matrices together has the combined effect. Note that we multiply PR because the rotation is applied first, and the projection second.

$$PR = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(∗∗) 15. If possible, compute the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Solution:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\begin{aligned}(A|I) &= \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right) \\ &\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right)\end{aligned}$$

Note that the last two rows of A after the first elimination step are scalar multiples of each other. Therefore A is *not* invertible. The reader may check that the matrix's determinant is zero.

(∗) 16. Show that $A = A^{-1} = A^T$, if $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. What is the effect of A acting on a 3×3 matrix?

Solution:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}(A|I) &= \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \\ &\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \\ &= (I|A^{-1})\end{aligned}$$

Thus $A = A^{-1}$

$$A^T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A$$

Thus all the equalities hold.

The effect of A acting on a 3×3 matrix is that it swaps the first two columns. Thus it makes sense that it is its own inverse. This type of matrix (with zeros everywhere and only a single 1 in each column) is called a permutation matrix, because it swaps (permutes) rows and columns when it multiplies another matrix from the left.

(**) 17. If $T : \mathbb{R}^2 \to \mathbb{R}^2$, is a linear transformation, and we know that

$$T\left(\begin{pmatrix}2\\3\end{pmatrix}\right) = \begin{pmatrix}5\\7\end{pmatrix}, \quad T\left(\begin{pmatrix}1\\4\end{pmatrix}\right) = \begin{pmatrix}-2\\3\end{pmatrix}$$

(a) Compute $T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right)$

(b) Find the matrix for the linear transformation T

(c) Find the inverse transformation T^{-1}

Solution:

(a)

$$\begin{aligned} T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) &= T\left(\begin{pmatrix}2-1\\3-4\end{pmatrix}\right) \\ &= T\left(\begin{pmatrix}2\\3\end{pmatrix}\right) - T\left(\begin{pmatrix}1\\4\end{pmatrix}\right) \\ &= \begin{pmatrix}5\\7\end{pmatrix} - \begin{pmatrix}-2\\3\end{pmatrix} \\ &= \begin{pmatrix}7\\4\end{pmatrix} \end{aligned}$$

Note: This is only possible due to the properties of a linear transformation, which permit addition/subtraction of input/outputs.

$$T\left(\begin{pmatrix}x_1 \pm x_2\\y_1 \pm y_2\end{pmatrix}\right) = T\left(\begin{pmatrix}x_1\\y_1\end{pmatrix}\right) \pm T\left(\begin{pmatrix}x_2\\y_2\end{pmatrix}\right)$$

(b) Using the property noted above, solve for the transformations of unit vectors to find the matrix for the linear transformation.

$$\begin{aligned} T\left(\begin{pmatrix}2\\3\end{pmatrix}\right) - 2T\left(\begin{pmatrix}1\\4\end{pmatrix}\right) &= T\left(\begin{pmatrix}0\\-5\end{pmatrix}\right) \\ &= \begin{pmatrix}5\\7\end{pmatrix} - 2\begin{pmatrix}-2\\3\end{pmatrix} \\ &= \begin{pmatrix}9\\1\end{pmatrix} \\ &= -5T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) \end{aligned}$$

$$T\left(\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$= \begin{pmatrix}-9/5\\-1/5\end{pmatrix}$$

$$\begin{aligned} T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) &= T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right)+\left(\begin{pmatrix}0\\1\end{pmatrix}\right) \\ &= \begin{pmatrix}7\\4\end{pmatrix}+\begin{pmatrix}-9/5\\-1/5\end{pmatrix} \\ &= \begin{pmatrix}26/5\\19/5\end{pmatrix}\end{aligned}$$

$$T\left(\begin{pmatrix}1\\0\end{pmatrix}\right)=\begin{pmatrix}26/5\\19/5\end{pmatrix}$$

$$T=\begin{pmatrix}26/5 & -9/5\\19/5 & -1/5\end{pmatrix}=\frac{1}{5}\begin{pmatrix}26 & -9\\19 & -1\end{pmatrix}$$

(c)

$$\begin{pmatrix}26/5 & -9/5\\19/5 & -1/5\end{pmatrix}^{-1}=\frac{1}{29}\begin{pmatrix}-1 & 9\\-19 & 26\end{pmatrix}$$

Note: If inverse transformation exists ($\det(A)\neq 0$), T^{-1} for a 2×2 matrix is:

$$T^{-1} = \begin{pmatrix}a & b\\c & d\end{pmatrix}^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d & -b\\-c & a\end{pmatrix}$$

(**) 18. If possible, compute the inverse of the matrix $A = \begin{pmatrix} 1 & -2 & 3 \\ 3 & 5 & 1 \\ 6 & 4 & 2 \end{pmatrix}$

Solution:

$$\begin{aligned}
(A|I) &= \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 3 & 5 & 1 & 0 & 1 & 0 \\ 6 & 4 & 2 & 0 & 0 & 1 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 11 & -8 & -3 & 1 & 0 \\ 0 & 16 & -16 & -6 & 0 & 1 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 11 & -8 & -3 & 1 & 0 \\ 0 & 1 & -1 & -6/16 & 0 & 1/16 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -6/16 & 0 & 1/16 \\ 0 & 11 & -8 & -3 & 1 & 0 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3/8 & 0 & 1/16 \\ 0 & 0 & 3 & -3+33/8 & 1 & -11/16 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3/8 & 0 & 1/16 \\ 0 & 0 & 1 & 3/8 & 1/3 & -11/48 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -1/8 & -1 & 11/16 \\ 0 & 1 & 0 & 0 & 1/3 & -1/6 \\ 0 & 0 & 1 & 3/8 & 1/3 & -11/48 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/8 & -1/3 & -17/48 \\ 0 & 1 & 0 & 0 & 1/3 & -1/6 \\ 0 & 0 & 1 & 3/8 & 1/3 & -11/48 \end{array}\right) \\
&= (I|A^{-1})
\end{aligned}$$

Thus

$$A^{-1} = \begin{pmatrix} -1/8 & -1/3 & 17/48 \\ 0 & 1/3 & -1/6 \\ 3/8 & 1/3 & -11/48 \end{pmatrix}$$

$$= \frac{1}{48\begin{pmatrix} -6 & -16 & 17 \\ 0 & 16 & -8 \\ 18 & 16 & -11 \end{pmatrix}}$$

(∗∗) 19. If A is an $n \times n$ matrix, and $\det(A) = x$, what are

(a) $\det(3A)$

(b) $\det(-A)$

(c) $\det(A^2)$

(d) $\det(A^{-1})$

Solution: All 4 parts utilize the knowledge of properties of determinants.

(a) A property of determinants of matrices is $\det(\text{cA}) = c^n \det(\text{A})$, so $\det(3A) = 3^n \cdot x$

(b) $\det(-A) = (-1)^n \cdot x$

(c) $\det(A^2) = x^2$

(d) $\det(A^{-1}) = x^{-1}$

(∗∗) 20. (a) Find the matrix $R : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects vectors across the line $y = -2x$.

(b) Show that $R^2 = I$.

(c) Reflect the vector $(-2, 3)$ across the line $y = -2x$.

Solution:

(a) Let $\mathbf{x}$ be the direction of the line. Then,

$$\mathbf{x} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

The formula for the reflection matrix is as follows:

$$\begin{aligned}
R &= 2\frac{\mathbf{x}\mathbf{x}^T}{\mathbf{x}^T\mathbf{x}} - I \\
&= \frac{\begin{pmatrix} -1 \\ 2 \end{pmatrix}\begin{pmatrix} -1 & 2 \end{pmatrix}}{\begin{pmatrix} -1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= 2\frac{\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}}{5} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2/5 & -4/5 \\ -4/5 & 8/5 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{pmatrix}
\end{aligned}$$

(b) We perform the straightforward computation

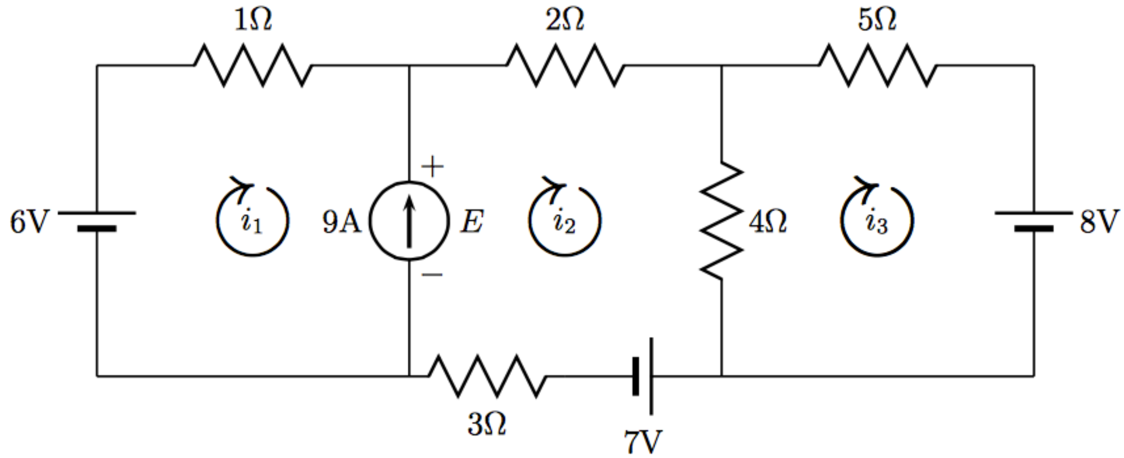
$$\begin{aligned}
R^2 &= \begin{pmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{pmatrix} \begin{pmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{pmatrix} \\
&= \frac{1}{25} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \\
&= \frac{1}{25} \begin{pmatrix} 3(3)+4(4) & 3(4)+4(-3) \\ 4(3)+(-3)(4) & 4(4)+(-3)(-3) \end{pmatrix} \\
&= \frac{1}{25} \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Intuitively, $R^2 = I$ because if you reflect twice, you get back to where you started, so there is no net change.

(c) Now reflecting the given vector across the given line,

$$\begin{aligned}
R\left(\begin{pmatrix} -2 \\ 3 \end{pmatrix}\right) &= \begin{pmatrix} -3/5 & -4/5 \\ -4/5 & 3/5 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} -3 & -4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} -3(-2)+(-4)(3) \\ -4(-2)+3(3) \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} -6 \\ 17 \end{pmatrix}
\end{aligned}$$

(∗∗) 21. Set up the augmented matrix A corresponding to this resistor network, with the loop currents in the first three columns.



Solution: To find the equations from each loop, go in the direction of the arrow (same for all loops) and sum up voltage drops (hence why going counter-clockwise through E is negative) and using a difference of currents when going through resistors bounded by two current loops.
The four equations are

$$\begin{aligned} -6 + 1(i_1) + E &= 0 \\ -E + 2(i_2) + 4(i_2 - i_3) + 7 + 3(i_2) &= 0 \\ 4(i_3 - i_2) + 5(i_3) + 8 &= 0 \\ i_2 - i_1 &= 9 \end{aligned}$$

Rearranging and simplifying:

$$\begin{aligned} i_1 + E &= 6 \\ 9i_2 - 4i_3 - E &= -7 \\ -4i_2 + 9i_3 &= -8 \\ -i_1 + i_2 &= 9 \end{aligned}$$

We thus have the matrix:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 6 \\ 0 & 9 & -4 & -1 & -7 \\ 0 & -4 & 9 & 0 & -8 \\ -1 & 1 & 0 & 0 & 9 \end{array}\right)$$

(∗∗) 22. If each year, 1/10 of electrical engineering students transfer to computer engineering, and 2/10 of computer engineering students transfer to electrical engineering, and there are initially 400 people in electrical engineering, and 600 people in computer engineering

(a) Find the transition matrix P

(b) Find how many students there are in each discipline after 2 years?

Solution:

(a) Suppose E is the number of electrical engineering students, and C is the number of computer engineering students. The probability transition matrix will obey the following relationship:

$$\begin{pmatrix} E \\ C \end{pmatrix}_{k+1} = P \begin{pmatrix} E \\ C \end{pmatrix}_{k}$$

From this we can discern the matrix P:

$$P = \begin{pmatrix} 9/10 & 2/10 \\ 1/10 & 8/10 \end{pmatrix}$$

and the initial condition

$$\begin{pmatrix} E \\ C \end{pmatrix}_{0} = \begin{pmatrix} 400 \\ 600 \end{pmatrix}$$

(b) We perform the computation:

- Students after the first step:

$$\begin{aligned} \begin{pmatrix} E \\ C \end{pmatrix}_{1} &= P \begin{pmatrix} E \\ C \end{pmatrix}_{0} \\ &= \begin{pmatrix} 9/10 & 2/10 \\ 1/10 & 8/10 \end{pmatrix} \begin{pmatrix} 400 \\ 600 \end{pmatrix} \\ &= \begin{pmatrix} 480 \\ 520 \end{pmatrix} \end{aligned}$$

- Students after the second step:

$$\begin{aligned} \begin{pmatrix} E \\ C \end{pmatrix}_{2} &= P \begin{pmatrix} E \\ C \end{pmatrix}_{1} \\ &= \begin{pmatrix} 9/10 & 2/10 \\ 1/10 & 8/10 \end{pmatrix} \begin{pmatrix} 480 \\ 520 \end{pmatrix} \\ &= \begin{pmatrix} 536 \\ 464 \end{pmatrix} \end{aligned}$$

(∗∗) 23. A Physics 158 course is taught in two sections, and initially 400 students are in section 201, and 350 students are in section 203. If every week 1/4 of those in section 201 and 1/3 of those in section 203 permanently drop the course, and 1/6 of each section transfer to the other section,

(a) Find the transition matrix P

(b) the number of students in each state after 2 weeks.

You may leave your answer in calculator ready form. (That is, there is no need to multiply out or add fractions to common denominators)

Solution:

(a) Let the number of students in section 201 be S_{201}, the number of students in section 203 be S_{203}, and the number of students who have dropped the course be D. The probability transition matrix will obey the following relation:

$$\begin{pmatrix} S_{201} \\ S_{203} \\ D \end{pmatrix}_{k+1} = P \begin{pmatrix} S_{201} \\ S_{203} \\ D \end{pmatrix}_{k} \Rightarrow P = \begin{pmatrix} 7/12 & 1/6 & 0 \\ 1/6 & 1/2 & 0 \\ 1/4 & 1/3 & 1 \end{pmatrix}$$

The initial condition will be

$$\begin{pmatrix} S_{201} \\ S_{203} \\ D \end{pmatrix}_{0} = \begin{pmatrix} 400 \\ 350 \\ 0 \end{pmatrix}$$

(b) We perform the computation:

- The number of students in each state after the first week will be:

$$\begin{aligned} \begin{pmatrix} S_{201} \\ S_{203} \\ D \end{pmatrix}_{1} &= P \begin{pmatrix} S_{201} \\ S_{203} \\ D \end{pmatrix}_{0} \\ &= \begin{pmatrix} 7/12 & 1/6 & 0 \\ 1/6 & 1/2 & 0 \\ 1/4 & 1/3 & 1 \end{pmatrix} \begin{pmatrix} 400 \\ 350 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 875/3 \\ 725/3 \\ 650/3 \end{pmatrix} \end{aligned}$$

- And the number of students in each state after the second week will be

$$\begin{aligned} \begin{pmatrix} S_{201} \\ S_{203} \\ D \end{pmatrix}_{2} &= P \begin{pmatrix} S_{201} \\ S_{203} \\ D \end{pmatrix}_{1} \\ &= \begin{pmatrix} 7/12 & 1/6 & 0 \\ 1/6 & 1/2 & 0 \\ 1/4 & 1/3 & 1 \end{pmatrix} \begin{pmatrix} 875/3 \\ 725/3 \\ 650/3 \end{pmatrix} \\ &= \begin{pmatrix} 2525/12 \\ 1525/9 \\ 5525/36 \end{pmatrix} \end{aligned}$$

Thus, after 2 weeks, the the number of students in each class is

$$\begin{pmatrix} S_{201} \\ S_{203} \\ D \end{pmatrix}_{2} = \begin{pmatrix} 2525/12 \\ 1525/9 \\ 5525/36 \end{pmatrix}$$

(**) 24. Given $A = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 1 & 5 & 8 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 1 & 5 & 9 \end{pmatrix}$

(a) Evaluate $\det(A)$ by reducing the matrix to upper triangular form.

(b) Compute the determinants of

i. B

ii. C

iii. AB

iv. $A^T A$

v. C^T

Solution:

(a)

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 1 & 5 & 8 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 4 & 5 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

Thus

$$\det(A) = (1)(4)(-1) = -4$$

(b) i.

$$\det(B) = (1)(4)(1) = 4$$

ii.

$$\det(C) = \begin{vmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 1 & 5 & 8 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{vmatrix} = -4 + 4 = 0$$

iii. $\det(AB) = (-4)(4) = -16$

iv. $\det(A^T A) = \det(A^2) = (-4)^2 = 16$

v. $\det(C^T) = \det(C) = 0$

(**) 25. Consider the linear system for the unknowns x, y, and z.

$$\begin{aligned} 4x + 2y - 3z - 6 &= 0 \\ x - 4y + z + 4 &= 0 \\ -x + 2z - 2 &= 0 \end{aligned}$$

(a) Write the system in an augmented matrix.

(b) Perform row operations on the augmented matrix to change it to upper triangular form.

(c) Find the solution to the problem from above.

Solution:

(a) Move constants to the right side of the equations.

$$A = \left(\begin{array}{ccc|c} 4 & 2 & -3 & 6 \\ 1 & -4 & 1 & -4 \\ -1 & 0 & 2 & 2 \end{array}\right)$$

(b)

$$\begin{aligned} A &= \left(\begin{array}{ccc|c} 4 & 2 & -3 & 6 \\ 1 & -4 & 1 & -4 \\ -1 & 0 & 2 & 2 \end{array}\right) \\ &\sim \left(\begin{array}{ccc|c} 4 & 2 & -3 & 6 \\ 0 & -\frac{9}{2} & \frac{7}{4} & -\frac{11}{2} \\ 0 & \frac{1}{2} & \frac{5}{4} & \frac{7}{2} \end{array}\right) \\ &\sim \left(\begin{array}{ccc|c} 4 & 2 & -3 & 6 \\ 0 & -18 & 7 & -22 \\ 0 & 2 & 5 & 14 \end{array}\right) \\ &\sim \left(\begin{array}{ccc|c} 4 & 2 & -3 & 6 \\ 0 & -18 & 7 & -22 \\ 0 & 0 & \frac{52}{9} & \frac{104}{9} \end{array}\right) \\ &\sim \left(\begin{array}{ccc|c} 4 & 2 & -3 & 6 \\ 0 & -18 & 7 & -22 \\ 0 & 0 & 52 & 104 \end{array}\right) \end{aligned}$$

(c) We can now apply back substitution to solve the system. We start with the last equation

$$52z = 104$$

to find

$$z = 2$$

Now plugging this result in to

$$-18y + 7z = -22$$

we can solve

$$y = 2$$

Now taking the results for y and z and plugging into the first equation,

$$4x + 2y - 3z = 6$$

we can find

$$x = 2$$

This yields the final result of

$$(x, y, z) = (2, 2, 2)$$

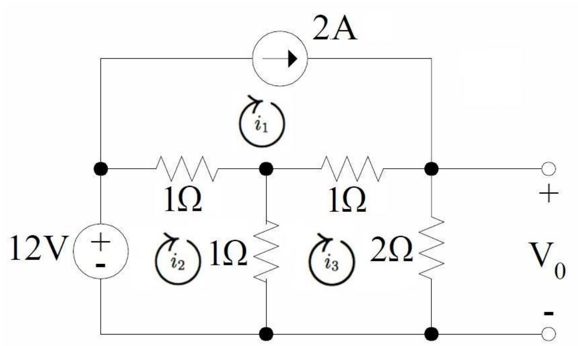
A unique solution to this linear system of equations.

(∗) 26. Consider the resistor network below:

(a) Set up the augmented matrix A corresponding to this resistor network

(b) Solve the augmented matrix for the currents 1,2, and 3

(c) Find the voltage V_0



Solution:

(a) The equations from the circuit are:

$$\begin{aligned} \textit{Loop } 1 &: \quad I_1 = 2 \\ \textit{Loop } 2 &: \quad -12 + (I_2 - I_1) + (I_2 - I_3) = 0 \\ &\Rightarrow \quad -I_1 + 2I_2 - I_3 = 12 \\ \textit{Loop } 3 &: \quad (I_3 - I_2) + (I_3 - I_1) + 2I_3 = 0 \\ &\Rightarrow \quad -I_1 - I_2 + 4I_3 = 0 \end{aligned}$$

Note: Loop 1 has a single current source inline with the direction of loop 1, hence the current of the loop must equal the current source 2A.
The first loop shows the current of loop 1 while the second and third loops use the theorem which states the voltages around a closed loop must sum to 0.
Then set up the augmented matrix:

$$A = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ -1 & 2 & -1 & 12 \\ -1 & -1 & 4 & 0 \end{array}\right)$$

(b) Then solve the augmented matrix using reduced row echelon form.

$$\begin{aligned}
A &= \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ -1 & 2 & -1 & 12 \\ -1 & -1 & 4 & 0 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 2 & -1 & 14 \\ 0 & -1 & 4 & 2 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 2 & -1 & 14 \\ 0 & 0 & \frac{7}{2} & 9 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 2 & -1 & 14 \\ 0 & 0 & 1 & \frac{18}{7} \end{array}\right) \\
&\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & \frac{116}{7} \\ 0 & 0 & 1 & \frac{18}{7} \end{array}\right) \\
&\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & \frac{58}{7} \\ 0 & 0 & 1 & \frac{18}{7} \end{array}\right) \\
(I_1, I_2, I_3) &= \left(2, \frac{58}{7}, \frac{18}{7}\right)
\end{aligned}$$

(c) V_0 is simply $2I_3 = \frac{36}{7}V$