# Mathematics 152 Midterm 3 Review Package –

UBC Engineering Undergraduate Society

Attempt questions to the best of your ability. Problems are ranked in difficulty as (∗) for easy, (∗∗) for medium, and (∗ ∗ ∗) for difficult.

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| Solutions posted at: `https://ubcengineers.ca/tutoring` |

If you believe that there is an error in these solutions, or have any questions, comments, or suggestions regarding EUS Tutoring sessions, please e-mail us at: tutoring@ubcengineers.ca. If you are interested in helping with EUS tutoring sessions in the future or other academic events run by the EUS, please e-mail vpacademic@ubcengineers.ca.

Some of the problems in this package were not created by the EUS. Those problems originated from one of the following sources:

- Schuam's Outline of Matrix Operations; Richard Bronson
- Calculus 7th ed; James Stewart
- Linear Algebra; Sterling K. Berberian
- Linear Algebra and Its Applications 3rd ed; Gilbert Strang
- Linear Algebra and Matrix Theory; Robert Stoll

| Want a warm up? | Short on study time? | Want a challenge? |
|---|---|---|
| These are the easier problems | These cover most of the material | These are some tougher questions |
| 1, 2, 3 | 7, 9, 10, 11 | 10, 11, 12 |

## EUS Health and Wellness Study Tips

- **Eat Healthy**—Your body needs fuel to get through all of your long hours studying. You should eat a variety of food (not just a variety of ramen) and get all of your food groups in.
- **Take Breaks**—Your brain needs a chance to rest: take a fifteen minute study break every couple of hours. Staring at the same physics problem until your eyes go numb won't help you understand the material.
- **Sleep**—We have all been told we need 8 hours of sleep a night, university should not change this. Get to know how much sleep you need and set up a regular sleep schedule.

Good Luck!

(∗) 1. Compute the determinant of the matrix: $\begin{pmatrix} 2 & 6 & \log 2 & \pi^2 & e \\ 0 & 5 & 2 & 4 & \sqrt{5} \\ 0 & 0 & \pi & \sin(9) & 7 \\ 0 & 0 & 0 & -4 & 21 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$

**Solution:** The determinant of a triangular matrix is just the product of its diagonal entries.

$$\begin{vmatrix} 2 & 6 & \log 2 & \pi^2 & e \\ 0 & 5 & 2 & 4 & \sqrt{5} \\ 0 & 0 & \pi & \sin(9) & 7 \\ 0 & 0 & 0 & -4 & 21 \\ 0 & 0 & 0 & 0 & 6 \end{vmatrix} = (2)(5)(\pi)(-4)(6) = -240\pi$$

(∗) 2. Compute the transpose of $A = \begin{pmatrix} -6 & 9 & 0 \\ 1 & -1 & 0 \\ 2 & \pi & 3 \\ 5 & 2 & 6 \end{pmatrix}$

**Solution:** To calculate the transpose of a matrix, simply swap the rows and columns (e.g. column 1 values of A going down become row 1 values going right).
With

$$A = \begin{pmatrix} -6 & 9 & 0 \\ 1 & -1 & 0 \\ 2 & \pi & 3 \\ 5 & 2 & 6 \end{pmatrix}$$

We compute

$$A^T = \begin{pmatrix} -6 & 1 & 2 & 5 \\ 9 & -1 & \pi & 2 \\ 0 & 0 & 3 & 6 \end{pmatrix}$$

(∗) 3. (a) Compute the product $A\mathbf{x} = \begin{pmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

(b) Without computing the determinant, determine if the matrix $A$ is invertible or not.

**Solution:**

(a) $A\mathbf{x} = \begin{pmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3(2) + (-6)(1) + 0(1) \\ 0(2) + 2(1) + (-2)(1) \\ 1(2) + (-1)(1) + (-1)(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Any multiple of $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ would also produce the zero vector.

(b) The matrix is not invertible because it maps nonzero vectors to the zero vector.

(∗) 4. Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 8 \\ 0 & -9 \end{pmatrix}$,

(a) If it is defined, compute $AB$

(b) If it is defined, compute $BA$

**Solution:**

(a) Since the matrix dimensions don't match for multiplication,

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{pmatrix} \begin{pmatrix} 7 & 8 \\ 0 & -9 \end{pmatrix} = \text{undefined}$$

(b) Since the matrix dimensions match for multiplication,

$$\begin{aligned} BA &= \begin{pmatrix} 7 & 8 \\ 0 & -9 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 7(1) + 8(4) & 7(2) + 8(-5) & 7(3) + 8(6) \\ 0(1) + (-9)(4) & 0(2) + (-9)(-5) & 0(3) + (-9)(6) \end{pmatrix} \\ &= \begin{pmatrix} 39 & -26 & 69 \\ -36 & 45 & -54 \end{pmatrix} \end{aligned}$$

(**) 5. If possible, compute the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

**Solution:**

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\begin{aligned} (A|I) &= \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right) \\ &\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right) \end{aligned}$$

Note that the last two rows of $A$ after the first elimination step are scalar multiples of each other. Therefore $A$ is *not* invertible. The reader may check that the matrix's determinant is zero.

(*) 6. Show that $A = A^{-1} = A^T$, if $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. What is the effect of $A$ acting on a $3 \times 3$ matrix?

**Solution:**

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} (A|I) &= \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \\ &\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \\ &= (I|A^{-1}) \end{aligned}$$

Thus $A = A^{-1}$

$$A^T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A$$

Thus all the equalities hold.

The effect of $A$ acting on a $3 \times 3$ matrix is that it swaps the first two columns. Thus it makes sense that it is its own inverse. This type of matrix (with zeros everywhere and only a single 1 in each column) is called a permutation matrix, because it swaps (permutes) rows and columns when it multiplies another matrix from the left.

(**) 7. If $T : \mathbb{R}^2 \to \mathbb{R}^2$, is a linear transformation, and we know that

$$T\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 7 \end{pmatrix}, \quad T\left(\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right) = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

(a) Compute $T\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$

(b) Find the matrix for the linear transformation $T$

(c) Find the inverse transformation $T^{-1}$

**Solution:**

(a)

$$\begin{aligned} T\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) &= T\left(\begin{pmatrix} 2-1 \\ 3-4 \end{pmatrix}\right) \\ &= T\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) - T\left(\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right) \\ &= \begin{pmatrix} 5 \\ 7 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ 4 \end{pmatrix} \end{aligned}$$

Note: This is only possible due to the properties of a linear transformation, which permit addition/subtraction of input/outputs.

$$T\left(\begin{pmatrix} x_1 \pm x_2 \\ y_1 \pm y_2 \end{pmatrix}\right) = T\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) \pm T\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right)$$

(b) Using the property noted above, solve for the transformations of unit vectors to find the matrix for the linear transformation.

$$\begin{aligned} T\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) - 2T\left(\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right) &= T\left(\begin{pmatrix} 0 \\ -5 \end{pmatrix}\right) \\ &= \begin{pmatrix} 5 \\ 7 \end{pmatrix} - 2\begin{pmatrix} -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ 1 \end{pmatrix} \\ &= -5T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \end{aligned}$$

$$T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} -9/5 \\ -1/5 \end{pmatrix}$$

$$\begin{aligned} T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) &= T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) + \left(\begin{pmatrix}0\\1\end{pmatrix}\right) \\ &= \begin{pmatrix}7\\4\end{pmatrix} + \begin{pmatrix}-9/5\\-1/5\end{pmatrix} \\ &= \begin{pmatrix}26/5\\19/5\end{pmatrix} \end{aligned}$$

$$T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}26/5\\19/5\end{pmatrix}$$

$$T = \begin{pmatrix}26/5 & -9/5\\19/5 & -1/5\end{pmatrix} = \frac{1}{5}\begin{pmatrix}26 & -9\\19 & -1\end{pmatrix}$$

(c)

$$\begin{pmatrix}26/5 & -9/5\\19/5 & -1/5\end{pmatrix}^{-1} = \frac{1}{29}\begin{pmatrix}-1 & 9\\-19 & 26\end{pmatrix}$$

Note: If inverse transformation exists ($\det(A) \neq 0$), $T^{-1}$ for a $2 \times 2$ matrix is:

$$T^{-1} = \begin{pmatrix}a & b\\c & d\end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d & -b\\-c & a\end{pmatrix}$$

(∗∗) 8. If possible, compute the inverse of the matrix $A = \begin{pmatrix} 1 & -2 & 3 \\ 3 & 5 & 1 \\ 6 & 4 & 2 \end{pmatrix}$

**Solution:**

$$\begin{aligned}
(A|I) &= \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 3 & 5 & 1 & 0 & 1 & 0 \\ 6 & 4 & 2 & 0 & 0 & 1 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 11 & -8 & -3 & 1 & 0 \\ 0 & 16 & -16 & -6 & 0 & 1 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 11 & -8 & -3 & 1 & 0 \\ 0 & 1 & -1 & -6/16 & 0 & 1/16 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -6/16 & 0 & 1/16 \\ 0 & 11 & -8 & -3 & 1 & 0 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3/8 & 0 & 1/16 \\ 0 & 0 & 3 & -3+33/8 & 1 & -11/16 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3/8 & 0 & 1/16 \\ 0 & 0 & 1 & 3/8 & 1/3 & -11/48 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -1/8 & -1 & 11/16 \\ 0 & 1 & 0 & 0 & 1/3 & -1/6 \\ 0 & 0 & 1 & 3/8 & 1/3 & -11/48 \end{array}\right) \\
&\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/8 & -1/3 & -17/48 \\ 0 & 1 & 0 & 0 & 1/3 & -1/6 \\ 0 & 0 & 1 & 3/8 & 1/3 & -11/48 \end{array}\right) \\
&= (I|A^{-1})
\end{aligned}$$

Thus

$$A^{-1} = \begin{pmatrix} -1/8 & -1/3 & 17/48 \\ 0 & 1/3 & -1/6 \\ 3/8 & 1/3 & -11/48 \end{pmatrix}$$

$$= 1 \frac{}{48 \begin{pmatrix} -6 & -16 & 17 \\ 0 & 16 & -8 \\ 18 & 16 & -11 \end{pmatrix}}$$

(**) 9. If $A$ is an $n \times n$ matrix, and $\det(A) = x$, what are

(a) $\det(3A)$

(b) $\det(-A)$

(c) $\det(A^2)$

(d) $\det(A^{-1})$

**Solution:** All 4 parts utilize the knowledge of properties of determinants.

(a) A property of determinants of matrices is $\det(\text{cA}) = c^n \det(\text{A})$, so $\det(3A) = 3^n \cdot x$

(b) $\det(-A) = (-1)^n \cdot x$

(c) $\det(A^2) = x^2$

(d) $\det(A^{-1}) = x^{-1}$

(**) 10. If each year, 1/10 of electrical engineering students transfer to computer engineering, and 2/10 of computer engineering students transfer to electrical engineering, and there are initially 400 people in electrical engineering, and 600 people in computer engineering

(a) Find the transition matrix $P$

(b) Find how many students there are in each discipline after 2 years?

**Solution:**

(a) Suppose $E$ is the number of electrical engineering students, and $C$ is the number of computer engineering students. The probability transition matrix will obey the following relationship:

$$\begin{pmatrix} E \\ C \end{pmatrix}_{k+1} = P \begin{pmatrix} E \\ C \end{pmatrix}_k$$

From this we can discern the matrix $P$:

$$P = \begin{pmatrix} 9/10 & 2/10 \\ 1/10 & 8/10 \end{pmatrix}$$

and the initial condition

$$\begin{pmatrix} E \\ C \end{pmatrix}_0 = \begin{pmatrix} 400 \\ 600 \end{pmatrix}$$

(b) We perform the computation:

- Students after the first step:

$$\begin{aligned} \begin{pmatrix} E \\ C \end{pmatrix}_1 &= P \begin{pmatrix} E \\ C \end{pmatrix}_0 \\ &= \begin{pmatrix} 9/10 & 2/10 \\ 1/10 & 8/10 \end{pmatrix} \begin{pmatrix} 400 \\ 600 \end{pmatrix} \\ &= \begin{pmatrix} 480 \\ 520 \end{pmatrix} \end{aligned}$$

- Students after the second step:

$$\begin{aligned}
\begin{pmatrix} E \\ C \end{pmatrix}_2 &= P \begin{pmatrix} E \\ C \end{pmatrix}_1 \\
&= \begin{pmatrix} 9/10 & 2/10 \\ 1/10 & 8/10 \end{pmatrix} \begin{pmatrix} 480 \\ 520 \end{pmatrix} \\
&= \begin{pmatrix} 536 \\ 464 \end{pmatrix}
\end{aligned}$$

(**) 11. A Physics 158 course is taught in two sections, and initially 400 students are in section 201, and 350 students are in section 203. If every week 1/4 of those in section 201 and 1/3 of those in section 203 permanently drop the course, and 1/6 of each section transfer to the other section,

(a) Find the transition matrix $P$

(b) the number of students in each state after 2 weeks.

You may leave your answer in calculator ready form. (That is, there is no need to multiply out or add fractions to common denominators)

**Solution:**

(a) Let the number of students in section 201 be $S_{201}$, the number of students in section 203 be $S_{203}$, and the number of students who have dropped the course be $D$. The probability transition matrix will obey the following relation:

$$\begin{pmatrix} S_{201} \\ S_{203} \\ D \end{pmatrix}_{k+1} = P \begin{pmatrix} S_{201} \\ S_{203} \\ D \end{pmatrix}_{k} \Rightarrow P = \begin{pmatrix} 7/12 & 1/6 & 0 \\ 1/6 & 1/2 & 0 \\ 1/4 & 1/3 & 1 \end{pmatrix}$$

The initial condition will be

$$\begin{pmatrix} S_{201} \\ S_{203} \\ D \end{pmatrix}_{0} = \begin{pmatrix} 400 \\ 350 \\ 0 \end{pmatrix}$$

(b) We perform the computation:

- The number of students in each state after the first week will be:

$$\begin{aligned} \begin{pmatrix} S_{201} \\ S_{203} \\ D \end{pmatrix}_{1} &= P \begin{pmatrix} S_{201} \\ S_{203} \\ D \end{pmatrix}_{0} \\ &= \begin{pmatrix} 7/12 & 1/6 & 0 \\ 1/6 & 1/2 & 0 \\ 1/4 & 1/3 & 1 \end{pmatrix} \begin{pmatrix} 400 \\ 350 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 875/3 \\ 725/3 \\ 650/3 \end{pmatrix} \end{aligned}$$

- And the number of students in each state after the second week will be

$$\begin{aligned} \begin{pmatrix} S_{201} \\ S_{203} \\ D \end{pmatrix}_{2} &= P \begin{pmatrix} S_{201} \\ S_{203} \\ D \end{pmatrix}_{1} \\ &= \begin{pmatrix} 7/12 & 1/6 & 0 \\ 1/6 & 1/2 & 0 \\ 1/4 & 1/3 & 1 \end{pmatrix} \begin{pmatrix} 875/3 \\ 725/3 \\ 650/3 \end{pmatrix} \\ &= \begin{pmatrix} 2525/12 \\ 1525/9 \\ 5525/36 \end{pmatrix} \end{aligned}$$

Thus, after 2 weeks, the the number of students in each class is

$$\begin{pmatrix} S_{201} \\ S_{203} \\ D \end{pmatrix}_{2} = \begin{pmatrix} 2525/12 \\ 1525/9 \\ 5525/36 \end{pmatrix}$$

(∗∗) 12. Given $A = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 1 & 5 & 8 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 1 & 5 & 9 \end{pmatrix}$

(a) Evaluate $\det(A)$ by reducing the matrix to upper triangular form.

(b) Compute the determinants of

i. $B$

ii. $C$

iii. $AB$

iv. $A^T A$

v. $C^T$

**Solution:**

(a)

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 1 & 5 & 8 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 4 & 5 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

Thus

$$\det(A) = (1)(4)(-1) = -4$$

(b) i.

$$\det(B) = (1)(4)(1) = 4$$

ii.

$$\det(C) = \begin{vmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 1 & 5 & 8 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{vmatrix} = -4 + 4 = 0$$

iii. $\det(AB) = (-4)(4) = -16$

iv. $\det(A^T A) = \det(A^2) = (-4)^2 = 16$

v. $\det(C^T) = \det(C) = 0$