Mathematics 101 Midterm Review Package - **Solutions**

UBC Engineering Undergraduate Society

Problems are ranked in difficulty as ($*$) for easy, ($**$) for medium, and ($***$) for difficult. Note that sometimes difficulty can be subjective, so do not be discouraged if you are stuck on a ($*$) problem.

Solutions posted at: `http://ubcengineers.ca/tutoring/`

If you believe that there is an error in these solutions, or have any questions, comments, or suggestions regarding EUS Tutoring sessions, please e-mail us at: tutoring@ubcengineers.ca. If you are interested in helping with EUS tutoring sessions in the future or other academic events run by the EUS, please e-mail vpacademic@ubcengineers.ca.

The first 7 problems are review of high school material and are highly optional. They cover the basics of the different functions covered in high school.

Some of the problems in this package were not created by the EUS. Those problems originated from one of the following sources:

- Schuam's Outline of Calculus 2 ed; Ayres Jr., Frank
- Calculus – Early Transcendentals 7 ed; Stewart, James
- Calculus – 3 ed; Spivak, Michael
- Calculus Volume 1 2 ed; Apostol, Tom

Want a warm up?	Short on study time?	Want a challenge?
These are the easier problems	These cover most of the material	These are some tougher questions
1, 2, 4, 5, 7, 12, 26	2, 6, 9, 10, 11, 14, 21, 24, 29	17, 18, 22, 25, 28, 29, 30

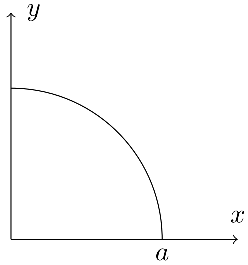
EUS Health and Wellness Study Tips

- **Eat Healthy**—Your body needs fuel to get through all of your long hours studying. You should eat a variety of food (not just a variety of ramen) and get all of your food groups in.
- **Take Breaks**—Your brain needs a chance to rest: take a fifteen minute study break every couple of hours. Staring at the same physics problem until your eyes go numb wont help you understand the material.
- **Sleep**—We have all been told we need 8 hours of sleep a night, university should not change this. Get to know how much sleep you need and set up a regular sleep schedule.

(*) 1. Compute the value of the integral in terms of a.

$$\int_0^a \sqrt{a^2 - x^2} dx$$

Solution:



We consider the area of the quarter circle of radius a.

$$\int_0^a \sqrt{a^2 - x^2} dx = \frac{\pi a^2}{4}$$

(*) 2. Suppose that

$$\int_{-2}^0 f(x)dx = 5, \quad \int_0^3 f(x)dx = -7, \quad \int_1^{-2} g(x)dx = -3, \quad \int_1^3 g(x) = 1$$

Find the value of

$$\int_{-2}^3 (f(x) - 2g(x))\, dx$$

Solution: By the following three properties of integrals:

$$\begin{aligned} \int_a^b f(x)dx + \int_b^c f(x)dx &= \int_a^c f(x)dx \\ \int_a^b f(x)dx + \int_a^b g(x)dx &= \int_a^b (f(x) + g(x))dx \\ -\int_a^b f(x)dx &= \int_b^a f(x)dx \end{aligned}$$

The integral $\int_{-2}^3 (f(x) - 2g(x))\, dx$ can be rewritten as follows:

$$\begin{aligned} \int_{-2}^3 (f(x) - 2g(x))\, dx &= \left(\int_{-2}^0 f(x)dx + \int_0^3 f(x)dx\right) - 2\left(-\int_1^{-2} g(x)dx + \int_1^3 g(x)dx\right) \\ &= (5 - 7) - 2(-(-3) + 1) \\ &= -10 \end{aligned}$$

(∗) 3. Compute the value of the integral.

$$\int_0^1 x\left(1-\sqrt{x}\right)^2 dx$$

Solution:

$$\begin{aligned}
\int_0^1 x(1-\sqrt{x})^2 dx &= \int_0^1 x(1-2\sqrt{x}+x)dx \\
&= \int_0^1 x - 2x^{3/2} + x^2 dx \\
&= \left(\frac{x^2}{2} - \frac{4x^{5/2}}{5} + \frac{x^3}{3}\right)\Bigg|_0^1 \\
&= \frac{1}{30}
\end{aligned}$$

$$\int_0^1 x(1-\sqrt{x})^2 dx = \frac{1}{30}$$

(∗) 4. Approximate the area under the graph of the function $y = x^2$ on the interval $(1,6)$ with 5 subdivisions using

(a) Right endpoints

(b) Left endpoints

Solution:

$$\Delta x = \frac{b-a}{n} = \frac{6-1}{5} = 1$$

(a) Right Endpoints:

$$\begin{aligned}
\sum_{i=1}^{5} f(x_i)\Delta x &= (1)\left(f(2)+f(3)+f(4)+f(5)+f(6)\right) \\
&= 4+9+16+25+36 \\
&= 90
\end{aligned}$$

(b) Left Endpoints:

$$\begin{aligned}
\sum_{i=0}^{4} f(x_i)\Delta x &= (1)\left(f(1)+f(2)+f(3)+f(4)+f(5)\right) \\
&= 1+4+9+16+25 \\
&= 55
\end{aligned}$$

(∗) 5. Evaluate the integral by using a limit of Riemann sums.

$$\int_0^4 -5x^2 + 7x - 2dx$$

The following summation formulas may be useful.

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(2n+1)(n+1)}{6}, \quad \sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Solution: We know that the value of the increment, Δx, is the length of the interval divided into n parts, so

$$\Delta x = \frac{4}{n}$$

This means we can write the x–coordinate of the ith point in the partition as

$$x_i = \frac{4i}{n}$$

Then evaluating the function $f(x) = -5x^2 + 7x - 2$ at the ith point in the partition, we obtain

$$\begin{aligned} f(x_i) &= -5\left(\frac{4i}{n}\right)^2 + 7\left(\frac{4i}{n}\right) - 2 \\ &= \frac{-80i^2}{n^2} + \frac{28i}{n} - 2 \end{aligned}$$

Now applying the definition of the integral via Riemann sums:

$$\begin{aligned} \int_0^4 -5x^2 + 7x - 2dx &= \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x \\ &= \lim_{x\to\infty} \frac{4}{n} \sum_{i=1}^n \left(\frac{-80i^2}{n^2} + \frac{28i}{n} - 2\right) \\ &= \lim_{x\to\infty} \frac{4}{n} \left(\frac{-80}{n^2} \sum_{i=1}^n i^2 + \frac{28}{n} \sum_{i=1}^n i - \sum_{i=1}^n 2\right) \\ &= \lim_{x\to\infty} \frac{4}{n} \left(\frac{-80}{n^2} \cdot \frac{n(2n+1)(n+1)}{6} + \frac{28}{n} \cdot \frac{n(n+1)}{2} - 2n\right) \\ &= \frac{-176}{3} \end{aligned}$$

(∗∗) 6. Consider the following Riemann sum:

$$\lim_{n\to\infty} \sum_{i=1}^n \sin^2\left(3 + \frac{i\pi}{n}\right) \frac{\pi}{n} = \int_a^b f(x)dx$$

Express this as a definite integral in two different ways. That is, provide two different sets of $a, b, f(x)$ such that the equality holds true.

Solution: From the properties of integrals that shifting both the interval and the function the same amount in opposite directions will result in the same exact calculated area, shown by the formula below:

$$\int_a^b f(x)dx = \int_{a-c}^{b-c} f(x+c)dx$$

First, the obvious integral from the standard approach will be:
1. Interval of integration is π seen from $\Delta x = \frac{\pi}{n}$.
2. $f(x_i) = \sin^2(3 + \frac{i\pi}{n})$ which shows that the interval begins at 3 at the partition where $i = 0$.
3. $f(x) = \sin^2(x)$ as the 3 is accounted for in the start of the interval.
Hence the first way to express the Riemann sum is:

$$\int_3^{3+\pi} \sin^2(x)dx$$

And using the property above, the second way to express the Riemann sum is:

$$\int_0^{\pi} \sin^2(x+3)dx$$

There are more ways to express the Riemann sum, but these two would be the simplest derivations.

(∗) 7. Write the following Riemann sum as a definite integral.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n} \sin\left(\sqrt{\log\left(2 + \frac{5i}{n}\right)}\right)$$

Solution: As the question is already prewritten in the recognizable form

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{5}{n}$ and $f(x_i) = \sin\left(\sqrt{\log\left(2 + \frac{5i}{n}\right)}\right)$
From this, we can see that the integral has a length of 5 as the because the term for Δx is the interval split up into nth partitions.
The term for the ith partition is $2 + \frac{5i}{n}$ which is a the same as the increment Δx with a multiple of i and begins at 2 where $i = 0$.
The function of the ith partition can be seen as $f(x) = \sin\left(\sqrt{\log(x)}\right)$.
Altogether:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n} \sin\left(\sqrt{\log\left(2 + \frac{5i}{n}\right)}\right) \quad = \quad \int_2^7 \sin\left(\sqrt{\log(x)}\right) dx$$

(∗∗) 8. Evaluate the integral.

$$\int \frac{e^{2x}}{1+e^{4x}}dx$$

Solution: For the integral

$$\int \frac{e^{2x}}{1+e^{4x}}dx$$

We make the substitution $u = e^{2x}$, and $du = 2e^{2x}dx$. The integral then becomes

$$\begin{aligned} \int \frac{e^{2x}}{1+e^{4x}}dx &= \frac{1}{2}\int \frac{1}{1+u^2}du \\ &= \frac{1}{2}\arctan u + C \\ &= \frac{1}{2}\arctan(e^{2x}) + C \end{aligned}$$

The final answer is then

$$\int \frac{e^{2x}}{1+e^{4x}}dx = \frac{1}{2}\arctan(e^{2x}) + C$$

(∗∗) 9. Evaluate the integral.

$$\int \frac{e^x-1}{e^x+1}dx$$

Solution:

$$\begin{aligned} \int \frac{e^x-1}{e^x+1}dx &= -\int \frac{1-e^x}{e^x+1}dx \\ &= -\int \frac{1+e^x}{e^x+1} - \frac{2e^x}{e^x+1}dx \\ &= -\int 1 - \frac{2e^x}{e^x+1}dx \\ &= -x + 2\int \frac{e^x}{e^x+1}dx \\ &= -x + 2\log(e^x+1) + C \\ &= 2\log(e^x+1) - x + C \end{aligned}$$

Thus

$$\int \frac{e^x-1}{e^x+1}dx = 2\log(e^x+1) - x + C$$

(∗∗) 10. Write the following Riemann sum as a definite integral.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right)$$

Solution: First, we notice that the form is already simplified by having $\frac{1}{n}$ factored out, but can be simplified further by factoring what remains like so:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right) \quad = \quad \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(1+\frac{i}{n}\right)^2$$

Then, we convert it into a definite integral by noticing two things:
First, the extracted $1/n$ term indicates an interval of 1 as that is the value of the increment Δx. Secondly, what remains is $f(x_i) = \left(1+\frac{i}{n}\right)^2$, so the function $f(x) = x^2$ beginning at $x = 1$. Finally, the Riemann sum can be written as a definite integral of this form:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right) \quad = \quad \int_1^2 x^2 dx$$

(∗∗) 11. Suppose you approximate the function $f(x) = \dfrac{x^4}{4} + x^2 + 5x - 7$ on the interval $[-1, 3]$ with a right endpoint Riemann sum. Will this approximation be an overestimate or an underestimate?

Solution: For a right Riemann sum, a positive slope will indicate an overestimation while a negative slope will indicate an underestimation.
Taking the first two derivative:

$$\begin{aligned} f(x) &= \frac{x^4}{4} + x^2 + 5x - 7 \\ f'(x) &= x^3 + 2x + 5 \\ f''(x) &= 3x^2 + 2 \end{aligned}$$

From the second derivative, we can see that the slope will always be increasing as $f''(x) = 3x^2 + 2$ will always be a positive term.
Next, we can see that the most negative value possible is from $x = -1$, which would put $f'(x) = (-1)^3 + 2(-1) + 5 = 2$ which is still positive, hence the slope is always positive on the interval from $[1, 3]$.
Altogether with the initial fact and the proof that the slope is positive on the interval, this approximation will be an overestimation.

(∗) 12. Compute the derivative of $f(x)$.

$$f(x) = \int_3^{x^2+5x} \sin t dt$$

Solution:

$$\begin{aligned} f'(x) &= (x^2+5x)'\sin(x^2+5x) \\ &= (2x+5)\sin(x^2+5x) \end{aligned}$$

(∗) 13. If $x(t) = e^t \cos t$ and $y(t) = e^t \sin t$, compute the value of the integral.

$$\int_2^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Solution: First we compute the derivatives x' and y':

$$x'(t) = e^t(\cos t - \sin t)$$
$$y'(t) = e^t(\sin t + \cos t)$$

Now plugging these results into the integral:

$$\begin{aligned} \int_2^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt &= \int_2^3 \sqrt{(e^t(\cos t - \sin t))^2 + (e^t(\sin t + \cos t))^2} dt \\ &= \int_2^3 \sqrt{e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t + \sin^2 t + 2\cos t \sin t + \cos^2 t)} dt \\ &= \int_2^3 \sqrt{2e^{2t}} dt \\ &= \int_2^3 e^t\sqrt{2} dt \\ &= \sqrt{2}\, e^t\Big|_2^3 \\ &= \sqrt{2}e^2(e-1) \end{aligned}$$

(∗∗) 14. Prove that

$$\int_0^{\pi/2} x \sin x dx \leq \frac{\pi^2}{8}$$

Solution: First we notice that we need to find something larger and simpler than the integral but is still smaller than $\frac{\pi^2}{8}$.
In this case, the case which stands out most is $0 \leq x \sin x \leq x$ on the interval of $[0, \frac{\pi}{2}]$. Thus, evaluating the new simpler integral below:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x dx &= \left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} \\ &= \frac{(\frac{x^2}{2})^2}{2} - \frac{(0)^2}{2} \\ &= \frac{\pi^2}{8} \end{aligned}$$

Since the larger integral is equal to $\frac{\pi^2}{8}$, the original integral must be smaller as:

$$\int_0^{\pi/2} x \sin x dx \leq \int_0^{\frac{\pi}{2}} x dx = \frac{\pi^2}{8}$$

(∗∗) 15. Compute the derivative of $f(x)$.

$$f(x) = \int_{\log(7x^2-2x+5)}^{9} \sqrt[4]{t^3 - \sin t} dt$$

Solution: First we split up the integral to rewrite f as follows:

$$\begin{aligned} f(x) &= \int_{\log(7x^2-2x+5)}^{9} \sqrt[4]{t^3 - \sin t} dt \\ &= -\int_{9}^{\log(7x^2-2x+5)} \sqrt[4]{t^3 - \sin t} dt \end{aligned}$$

Differentiating:

$$f'(x) = \frac{\sqrt[4]{(\log(7x^2 - 2x + 5))^3 - \sin(\log(7x^2 - 2x + 5))}}{14x - 2}$$

(∗∗) 16. Compute the derivative of $f(x)$.

$$f(x) = \int_{e^x}^{\sin(5x-9)} \arctan\left(\frac{1}{t}\right) dt$$

Solution: First we split the integral up and rewrite f as follows:

$$\begin{aligned} f(x) &= \int_{e^x}^{\sin(5x-9)} \arctan\left(\frac{1}{t}\right) dt \\ &= \int_{0}^{\sin(5x-9)} \arctan\left(\frac{1}{t}\right) dt - \int_0^{e^x} \arctan\left(\frac{1}{t}\right) dt \end{aligned}$$

Then differentiating:

$$f'(x) = 5\cos(5x - 9)\arctan(\csc(5x - 9)) - e^x \arctan(e^{-x})$$

(∗ ∗ ∗) 17. Evaluate the integral

$$\int_a^b e^x dx$$

using a Riemann sum. You may use L'Hopital's rule to evaluate any limits.

Hint 1. The following formula may be useful.

$$\sum_{k=0}^{n-1} z^k = \frac{z^n - 1}{z - 1}$$

Solution: First, write the integral as an infinite Riemann sum using two facts:

$$\begin{aligned} \Delta x &= \frac{b-a}{n} \\ f(x_i) &= e^{a+\frac{(b-a)i}{n}} = e^a \left(e^{\frac{b-a}{n}}\right)^i \end{aligned}$$

Then, put it together using the standard format:

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x$$

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \left(\frac{b-a}{n}\right) e^a \left(e^{\frac{b-a}{n}}\right)^i$$

And since the Δx has no i, it can be extracted out of the summation.

$$\lim_{n\to\infty} \left(\frac{b-a}{n}\right) e^a \sum_{i=0}^{n-1} e^{\frac{(b-a)i}{n}}$$

Then the hint can be used to simplify the Riemann sum into a single limit.

$$\begin{aligned} &= e^a(b-a) \lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{i=0}^{n-1} e^{\frac{(b-a)i}{n}} \\ &= e^a(b-a) \lim_{n\to\infty} \left(\frac{1}{n}\right) \left(\frac{\left(e^{\frac{b-a}{n}}\right)^n - 1}{\left(e^{\frac{b-a}{n}}\right) - 1}\right) \\ &= e^a(b-a)(e^{b-a} - 1) \lim_{n\to\infty} \frac{1}{ne^{\frac{b-a}{n}} - n} \\ &= e^a(b-a)(e^{b-a} - 1) \lim_{n\to\infty} \frac{1/n}{e^{\frac{b-a}{n}} - 1} \end{aligned}$$

Thus the limit is a $0/0$ case, it is an indeterminate form hence use L'Hopitale's Rule with respect to n.

$$\begin{aligned} &= e^a(b-a)(e^{b-a} - 1) \lim_{n\to\infty} \frac{-1/n^2}{e^{\frac{b-a}{n}}(-1/n^2)(b-a)} \\ &= e^a(e^{b-a} - 1) \lim_{n\to\infty} \frac{1}{e^{\frac{b-a}{n}}} \\ &= e^a(e^{b-a} - 1) \\ &= e^b - e^a \end{aligned}$$

(∗ ∗ ∗) 18. (a) Let A_n be the area of a polygon with n equal sides inscribed in a circle of radius r. Show that

$$A_n = \frac{1}{2}nr^2 \sin\left(\frac{2\pi}{n}\right)$$

(b) Evaluate the limit

$$\lim_{n\to\infty} A_n$$

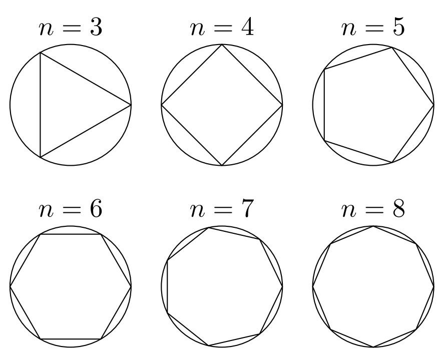
Solution:

(a) First, the formula would only be true for polygons inscribed with $n > 2$ for there to be any area. There are several points to notice about inscribed polygons in a circle:

- For each side (n) splits the angle occupied by each side by exactly that amount of $\frac{2\pi}{n}$
- Every side of a polygon can be cut into a separate triangle
- The height of the each separate triangle formed can be found using $r\sin\left(\frac{2\pi}{n}\right)$
- The area of each separate triangle is $\frac{1}{2}r^2\sin\left(\frac{2\pi}{n}\right)$

Compiling all points, the formula for the area would be $A_n = \frac{1}{2}nr^2\sin\left(\frac{2\pi}{n}\right)$

(b) Just from the images below, it can be seen that the polygon more closely resembles a circle for every increment of n until it becomes a circle itself, so $\lim_{n\to\infty} A_n = \pi r^2$.



(∗∗) 19. Evaluate the integral

$$\int_0^{\pi} e^{\cos t}\sin(2t)dt$$

Solution: First we will find the antiderivative. Let $u = \cos t$, and $du = -\sin t dt$.

$$\begin{aligned}\int e^{\cos t}\sin(2t)dt &= \int e^{\cos t} 2\sin t\cos t dt\\ &= -2\int e^u u du\\ &= -2(u-1)e^u + C\\ &= 2(1-\cos t)e^{\cos t} + C\end{aligned}$$

Now evaluating at the bounds of integration, we have

$$\int_0^{\pi} e^{\cos t}\sin(2t)dt = 2(1-\cos\pi)e^{\cos\pi} - 0 = \frac{4}{e}$$

(∗∗) 20. Evaluate the integral

$$\int \cos\left(\sqrt{x}\right) dx$$

Solution: First make the substitution $u = \sqrt{x}$, and $2udu = dx$:

$$\int \cos\left(\sqrt{x}\right) dx = \int \cos(u)2udu$$

Now integrating by parts, we have

$$\int \cos(u)2udu = 2\left(u\sin u - \int \sin udu\right)$$

Thus the final answer in terms of u is

$$2u\sin u + 2\cos u + C$$

Then putting back in terms of x,

$$\int \cos\left(\sqrt{x}\right) dx = 2\sqrt{x}\sin\left(\sqrt{x}\right) + 2\cos\left(\sqrt{x}\right) + C$$

(∗∗) 21. Evaluate the integral.

$$\int \frac{1}{1-\sin(x/2)}dx$$

Solution: We will need to multiply by the conjugate to solve this problem:

$$\begin{aligned}
\int \frac{1}{1-\sin(x/2)}dx &= \int \frac{1+\sin(x/2)}{1-\sin^2(x/2)}dx \\
&= \int \frac{1+\sin(x/2)}{\cos^2(x/2)}dx \\
&= \int \sec^2\left(\frac{x}{2}\right) + \tan\left(\frac{x}{2}\right)\sec\left(\frac{x}{2}\right) dx \\
&= 2\left(\tan\left(\frac{x}{2}\right) + \sec\left(\frac{x}{2}\right)\right) + C
\end{aligned}$$

The final answer is then

$$\int \frac{1}{1-\sin(x/2)}dx = 2\left(\tan\left(\frac{x}{2}\right) + \sec\left(\frac{x}{2}\right)\right) + C$$

(∗ ∗ ∗) 22. Evaluate the integral.

$$\int \frac{\cos 2x}{\sin^2(2x)+8}dx$$

Solution: For the integral

$$\int \frac{\cos 2x}{\sin^2(2x)+8}dx$$

We make the substitution $u = \sin 2x$, so the differential is $du = 2\cos 2x\, dx$.

$$\begin{aligned} \int \frac{\cos 2x}{\sin^2 2x+8}dx &= \frac{1}{2}\int \frac{1}{u^2+8}du \\ &= \frac{1}{16}\int \frac{1}{\left(\frac{u}{2\sqrt{2}}\right)^2+1}du \end{aligned}$$

Now with this integral, make the substitution $w = \frac{u}{2\sqrt{2}}$, and $du = 2\sqrt{2}dw$, to easily evaluate the integral:

$$\begin{aligned} \frac{1}{16}\int \frac{1}{\left(\frac{u}{2\sqrt{2}}\right)^2+1}du &= \frac{\sqrt{2}}{8}\int \frac{1}{1+w^2}dw \\ &= \frac{\sqrt{2}}{8}\arctan w + C \\ &= \frac{\sqrt{2}}{8}\arctan\left(\frac{\sin 2x}{2\sqrt{2}}\right)+C \end{aligned}$$

(∗ ∗ ∗) 23. Evaluate the integral

$$\int x \arctan x dx$$

Solution: First we need to compute

$$\int \arctan x dx$$

The way to do this is to integrate by parts, letting $u' = 1$, and $v = \arctan x$. Then we have

$$\int \arctan x dx = x\arctan x - \int \frac{x}{x^2+1}dx$$

This then evaluates to

$$\int \arctan x dx = x\arctan x - \frac{1}{2}\log(1+x^2)$$

Now we return to the original integral. We will have to integrate by parts. Let $x = u'$, and

$\arctan x = v$. Then we have

$$\begin{aligned}\int x \arctan x dx &= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx \\ &= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}dx \\ &= \frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{\arctan x}{2} + C\end{aligned}$$

Thus we have

$$\int x \arctan x dx = \frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{\arctan x}{2} + C$$

(**) 24. Evaluate the integral.

$$\int (\sin x)(\sin 3x)dx$$

Solution: We will have to integrate this by parts twice:

$$\begin{aligned}\int (\sin x)(\sin 3x)dx &= -\cos x \sin 3x + 3\int \cos x \cos 3x dx \\ &= -\cos x \sin 3x + 3\left(\sin x \cos 3x + 3\int \sin x \sin 3x dx\right) \\ &= -\cos x \sin 3x + 3\sin x \cos 3x + 9\int \sin x \sin 3x dx\end{aligned}$$

Rearranging:

$$-8\int \sin x \sin 3x dx = -\cos x \sin 3x + 3\sin x \cos 3x$$

$$\int \sin x \sin 3x dx = \frac{1}{8}\cos x \sin 3x - \frac{3}{8}\sin x \cos 3x + C$$

(* * *) 25. Evaluate the integral.

$$\int \sqrt{1-\cos x}dx$$

Solution: Use the double angle identity

$$1 - \cos 2x = 2\sin^2 x \Rightarrow 1 - \cos x = 2\sin^2\left(\frac{x}{2}\right)$$

$$\begin{aligned}\int \sqrt{1-\cos x}dx &= \int \sqrt{2\sin^2\left(\frac{x}{2}\right)}dx \\ &= \sqrt{2}\int \sin\left(\frac{x}{2}\right)dx \\ &= -2\sqrt{2}\cos\left(\frac{x}{2}\right)+C\end{aligned}$$

$$\int \sqrt{1-\cos x}dx = -2\sqrt{2}\cos\left(\frac{x}{2}\right)+C$$

(*) 26. Find the volume of the solid generated by revolving the plane area bounded by $x-y-7=0$, $x=9-y^2$ about the y-axis.

Solution:

$$\begin{aligned}V &= \int_{-2}^{1}\pi x_{\text{outer}}^2 - \pi x_{\text{inner}}^2 dy \\ &= \pi\int_{-2}^{1}(9-y^2)^2-(y+7)^2dy \\ &= \frac{333\pi}{5}\end{aligned}$$

(*) 27. Find the volume of the solid generated by revolving the plane area bounded by $y=x^2$ and $y=4x-x^2$ around the line $y=6$.

Solution: Since we are revolving about the line $y=6$, we will have to increase the radius of the disk in question.

$$\begin{aligned}V &= \int_0^2 \pi(6-y_{\text{outer}})^2-\pi(6-y_{\text{inner}})^2dx \\ &= \pi\int_0^2 12(y_{\text{inner}}-y_{\text{outer}})+y_{\text{outer}}^2-y_{\text{inner}}^2dx \\ &= \pi\int_0^2 12(4x-2x^2)+x^4-(4x-x^2)^2dx \\ &= \frac{64\pi}{3}\end{aligned}$$

Please see the figure for an illustration of the region being rotated.

(* * *) 28. Compute the area enclosed by the curve $y^2=x^2-x^4$.

Hint 2. *This curve is symmetric with respect to the x axis and the y axis. How can you use symmetry to help you calculate the area?*

Solution: See the figure which shows the area we need to compute.

First we make some observations as to how one would sketch the graph.

- First note that this graph is symmetric with respect to both the x axis and y axis, because replacing x by $-x$ or y by $-y$ leaves the relation unchanged. This means we can find the area enclosed in one quadrant, then multiply by 4.
- There are zeros at $x = 0$ and $x = 1$, which means we can draw an arc between those two points, then reflect it about the x and y axes for the total graph.
- The graph cannot extend beyond $x = 1$ because otherwise the expression inside square root that is obtained when solving for y would become negative. Solving for y,

$$y = x\sqrt{1-x^2}$$

  which holds for the portion in the first quadrant.

Integrating this expression from 0 to 1, then multiplying by 4 yields

$$4\int_0^1 x\sqrt{1-x^2}dx = 4\left(\frac{-1}{3}\right)\left(1-x^2\right)^{3/2}\Big|_0^1 = \frac{4}{3}$$

(∗∗∗) 29. A solid has a base in the form of an ellipse with major diameter 10 and minor diameter 8. Find the volume if every section perpendicular to the major axis is an isosceles triangle with altitude 6.

Solution: We will have to determine the area of the cross sectional triangle $A(x)$ as a function of the coordinate x along the ellipse. The width of the rectangle at position x will be $2y$, and the height will be 6, so the area is then $A(x) = (2y)(6)/2$ because it is an isoceles triangle. Thus we have

$$\begin{aligned}
V &= \int_{-5}^{5} A(x)dx \\
&= \int_{-5}^{5} 6ydx \\
&= \int_{-5}^{5} 6\left(\frac{4}{5}\sqrt{25-x^2}\right)dx \\
&= \frac{24}{5}\int_{-5}^{5}\sqrt{25-x^2}dx \\
&= \left(\frac{24}{5}\right)\left(\frac{5^2\pi}{2}\right) \\
&= 60\pi
\end{aligned}$$

(∗∗∗) 30. The base of a solid is the circle $x^2 + y^2 = 16x$, and every plane section perpendicular to the x-axis is a rectangle whose height is twice the distance of the plane of the section from the origin. Find the volume of this solid.

Solution: We will have to determine the area of the cross sectional rectangle $A(x)$ as a function of the coordinate x along the circle. The circle $x^2 + y^2 = 16x$ can be rewritten $(x-8)^2 + y^2 = 64$, by completing the square. Thus it is a circle of radius 8 centred at $(x, y) = (8, 0)$. The base of the rectangle will have length $2y$, and the height of the rectangle will be $2x$, so the area is then $A(x) = (2x)(2y) = 4xy$.

$$\begin{aligned}
V &= \int_0^{16} A(x)dx \\
&= \int_0^{16} 4xydx \\
&= \int_0^{16} 4x\sqrt{16x - x^2}dx \\
&= \int_0^{16} 4x\sqrt{16x - x^2}dx \\
&= \int_0^{16} 4x\sqrt{64 - (8-x)^2}dx
\end{aligned}$$

To evaluate the antiderivative, make the substitution $8 - x = u$, and $-du = dx$ to obtain

$$\begin{aligned}
\int_0^{16} 4x\sqrt{64 - (8-x)^2}dx &= 4\int_{-8}^{8} (8-u)\sqrt{64 - u^2}du \\
&= 4\left[8\int_{-8}^{8} \sqrt{64 - u^2}du - \int_{-8}^{8} u\sqrt{64 - u^2}du\right] \\
&= 4\left[8\frac{8^2\pi}{2} + \frac{1}{3}\left(64 - u^2\right)^{3/2}\Big|_{-8}^{8}\right] \\
&= 1024\pi
\end{aligned}$$