Physics 158 Midterm 2 Review Package

UBC Engineering Undergraduate Society

Problems are ranked in difficulty as (∗) for easy, (∗∗) for medium, and (∗ ∗ ∗) for difficult. Note that sometimes difficulty can be subjective, so do not be discouraged if you are stuck on a (∗) problem.

Solutions posted at: `https://ubcengineers.ca/tutoring`

If you believe that there is an error in these solutions, or have any questions, comments, or suggestions regarding EUS Tutoring sessions, please e-mail us at: tutoring@ubcengineers.ca. If you are interested in helping with EUS tutoring sessions in the future or other academic events run by the EUS, please e-mail vpacademic@ubcengineers.ca.

Want a warm up? These are the easier problems	Short on study time? These cover most of the material	Want a challenge? These are some tougher questions
1, 2, 3	3,4,5	7, 8,9,10

Some of the problems in this package were not created by the EUS. Those problems originated from one of the following sources:

- Introduction to Electrodynamics 3 ed. / David J. Griffths
- Electricity, Magnetism, and Light / Wayne Saslow
- Exercises for the Feynman Lectures on Physics / Matthew Sands, Richard Feynman, Robert Leighton.

All solutions prepared by the EUS.

EUS Health and Wellness Study Tips

- **Eat Healthy**—Your body needs fuel to get through all of your long hours studying. You should eat a variety of food (not just a variety of ramen) and get all of your food groups in.
- **Take Breaks**—Your brain needs a chance to rest: take a fifteen minute study break every couple of hours. Staring at the same physics problem until your eyes go numb wont help you understand the material.
- **Sleep**—We have all been told we need 8 hours of sleep a night, university should not change this. Get to know how much sleep you need and set up a regular sleep schedule.

Good Luck!

(∗) 1. An AC generator produces 12 A rms at 400 V rms with power factor 1.

(a) Find the rms power produced by this generator.

(b) The generator voltage gets boosted by a step-up transformer to 12 kV. Find the power after the step-up transformer, assuming no losses in the transformer.

(c) The power is then transmitted to an electrical load with wires having resistance 8 Ω each way, until it reaches a step-down transformer. Determine the rms power loss in the wires.

(d) Determine the power available to the load.

Solution:

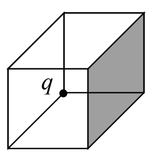
(a) Since the power factor is 1, $\cos\phi = 1$, so $\phi = 0$. Thus the average power dissipated by the circuit (over 1 period) is $P = I_{\text{rms}}V_{\text{rms}} = (12)(400) = 4800$ J.

(b) Since the transformer is lossless, the power is still $P = 4800$ J.

(c) The rms current *after* passing through the transformer is given by $VI = VI$, so $(12)(400) = (12000)I$, so $I = 0.4$ A. Using the formula $P = I_{\text{rms}}^2 R$, we have $P = (0.4^2)(8 + 8) = 2.56$ J.

(d) The power available will be $4800 - 2.56 = 4797.44$ J.

(∗) 2. Consider a charge q located at the corner of a cube. Find the electric flux through the indicated side.

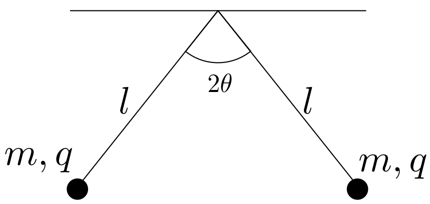


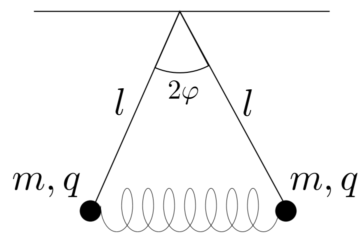
Solution: Consider a larger cube composed of eight of these smaller cubes shown, with q at the centre. The electric flux through this large cube will be q/ϵ_0. The indicated side is 1/24 the area of the total cube, so it has flux $\Phi_E = q/24\epsilon_0$ through it.

(∗∗) 3. Suppose we have two masses with equal mass m and equal charges $+q$ on them. They are hanging from the ceiling by massless rods of length l. Let $0 < 2\theta < \pi$ be the angle between the two rods at equilibrium, when the masses are attached only to their respective rods. Suppose we want the two rods to be separated by an angle $2\varphi < 2\theta$.

(a) If an (insulating) spring of *initial* length $l_0 = l \sin\varphi$ connects the two masses, what spring constant k must the spring have in order to keep the angle between the two rods 2φ?

(b) If we introduce a uniform downward pointing electric field of strength E_0, what now is the spring constant required to keep the angle between the two rods 2φ?





Solution:

(a) When the spring is introduced, the system will reach an equilibrium, where for an appropriate choice of k, the distance between the two masses is $2l \sin\varphi$. This means that the spring will be stretched by $l \sin\varphi$ from its initial length. Let's consider the forces on the left particle. If the tension in the rod is T, the force equilibrium in the y direction is

$$F_y = T\cos\varphi - mg = 0$$

Which yields

$$T = \frac{mg}{\cos\varphi}$$

The force equilibrium in the x direction is

$$F_x = T\sin\varphi - \frac{q^2}{4\pi\epsilon_0(2l\sin\varphi)^2} + k\Delta x = 0$$

Since $\Delta x = l\sin\varphi$, we can plug in the value for T found above and find k

$$k = \frac{q^2}{16\pi\epsilon_0 l^3 \sin^3\varphi} - \frac{mg}{l\cos\varphi}$$

(b) If we introduce a downward pointing electric field, we will have a different tension in the rod because the field will push the positive charges downward. Let's again consider the left particle. We can calculate this as

$$\sum F_y = T\cos\varphi - mg - E_0 q = 0$$

This means that the new tension is

$$T = \frac{mg + E_0 q}{\cos\varphi}$$

The force equilibrium in the x direction will remain the same, and thus the spring constant will be

$$k = \frac{q^2}{16\pi\epsilon_0 l^3 \sin^3\varphi} - \frac{E_0 q + mg}{l\cos\varphi}$$

(∗∗) 4. Consider an RLC circuit with $L = 2.5$ mH, $R = 4\ \Omega$, $C = 500\ \mu$F, driven by an AC voltage source of amplitude 24 V, and frequency 400 Hz.

(a) Find X_L, X_C, Z, and ϕ.

(b) Find the maximum current and maximum voltage across each of L, C, R.

(c) Find the time by which the driving voltage leads (or lags) the current.

(d) Find the maximum voltage across the combination R and C, and the time by which this voltage leads (or lags) the current.

(e) Find the maximum voltage across the combination R and L, and the time by which this voltage leads (or lags) the current.

(f) Find the maximum voltage across the combination L and C, and the time by which this voltage leads (or lags) the current.

Solution:

(a) First, the angular frequency of the circuit is $\omega = 800\pi$. Thus we have $X_L = \omega L = 2\pi = 6.28$ Ω, $X_C = 1/\omega C = 2.5/\pi = 0.796\ \Omega$, $Z = \sqrt{(X_L - X_C)^2 + R^2} = 6.79\ \Omega$, and $\phi = \arctan((X_L - X_C)/R) = 53.9°$.

(b) The maximum current through each of R, C, L will be the same for each element. This current will be

$$I_0 = \frac{V_0}{Z} = \frac{24}{\sqrt{(X_L - X_C)^2 + R^2}} = \frac{24}{\sqrt{6.79}} = 3.53 \text{ A}$$

The maximum voltage across each element will be its reactance multiplied by I_0. Thus,

- $V_{\text{max L}} = X_L I_0 = (6.28)(3.53) = 22.2$ V
- $V_{\text{max C}} = X_C I_0 = (0.796)(3.53) = 2.81$ V
- $V_{\text{max R}} = RI_0 = (4)(3.53) = 14.13$

(c) Since the current lags the driving voltage by phase angle ϕ, we have $\phi = \omega t$, if ϕ is radians. Thus, $0.94 = (800\pi)(t) \Rightarrow t = 3.74 \cdot 10^{-4}$. So the current *lags* the voltage by $3.74 \cdot 10^{-4}$ s. Or, equivalently, the voltage *leads* the current by $3.74 \cdot 10^{-4}$ s.

(d) Since the voltages across R and C are 90° out of phase, to find the maximum voltage across their combination requires us to add the voltages across them in quadrature. Thus we have $V_{\text{max C,R}} = \sqrt{V_{\text{max C}}^2 + V_{\text{max R}}^2} = 14.41$ V. The time by which the voltage across this combination will *lag* the current is given by $\varphi = \omega t$, where $\varphi = \arctan(X_C/R)$. Thus $t = 7.81 \cdot 10^{-5}$ s. (Note that φ must be in radians)

(e) Since the voltages across R and L are 90° out of phase, to find the maximum voltage across their combination requires us to add the voltages across them in quadrature. Thus we have $V_{\text{max L,R}} = \sqrt{V_{\text{max L}}^2 + V_{\text{max R}}^2} = 26.3$ V. The time by which the voltage across this combination will *lead* the current is given by $\varphi = \omega t$, where $\varphi = \arctan(X_L/R)$. Thus $t = 3.99 \cdot 10^{-4}$ s. (Note that φ must be in radians)

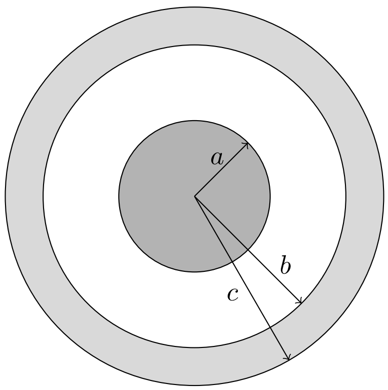
(f) Since the voltages across R and L are 180°out of phase, to find the maximum voltage across their combination requires us to subtract the max voltages across them. Thus we have $V_{\text{max C,L}} = V_{\text{max L}} - V_{\text{max C}} = 22.2 - 2.81 = 19.39$ V. The angle by which the max voltage across them *leads* the current is 90°, which means that we have $\varphi = \omega t = \pi/2 = 800\pi t$. Thus $t = 6.25 \cdot 10^{-4}$ s

(**) 5. Suppose we have a long solid insulating cylinder of radius a with volume charge density $\rho(r) = \rho_0(1-r/a)$ (in C/m^3), and a long concentric conducting shell of inner radius $b > a$ and outer radius $c > b$. There is a net linear charge density of λ (in C/m) on the conducting shell. See the cross section below.

(a) Calculate the electric field as a function of r.

(b) Calculate the linear surface charge density σ_b (in C/m) on the inner surface of the conducting shell (at radius b).

(c) Calculate the linear surface charge density σ_c (in C/m) on the outer surface of the conducting shell (at radius c).



Solution:

(a) First we need to calculate the electric field inside the insulating cylinder ($r < a$). Using Gauss's Law on an imaginary cylindrical surface of length l and radius r, and using the dummy variable s, we find

$$\begin{aligned}
\int \mathbf{E} \cdot d\mathbf{A} &= \frac{Q}{\epsilon_0} \\
2\pi r l E &= \frac{1}{\epsilon_0} \int_0^r \rho(s)(2\pi s l) ds \\
&= \frac{2\pi l \rho_0}{\epsilon_0} \int_0^r \left(1 - \frac{s}{a}\right) s ds \\
&= \frac{2\pi l \rho_0}{\epsilon_0} \int_0^r \left(s - \frac{s^2}{a}\right) ds \\
&= \frac{2\pi l \rho_0}{\epsilon_0} \left(\frac{r^2}{2} - \frac{r^3}{3a}\right)
\end{aligned}$$

Now solving for E, we have

$$E(r) = \frac{\rho_0}{\epsilon_0}\left(\frac{r}{2} - \frac{r^2}{3a}\right), \quad r < a$$

The electric field for $a < r < b$ can also be calculated with Gauss's Law.

$$\begin{aligned}
\int \mathbf{E}\cdot d\mathbf{A} &= \frac{Q}{\epsilon_0} \\
2\pi r l E &= \frac{1}{\epsilon_0}\int_0^a \rho(s)(2\pi s l)ds \\
&= \frac{2\pi l \rho_0}{\epsilon_0}\int_0^a \left(1-\frac{s}{a}\right) s ds \\
&= \frac{2\pi l \rho_0}{\epsilon_0}\int_0^a \left(s-\frac{s^2}{a}\right) ds \\
&= \frac{2\pi l \rho_0}{\epsilon_0}\frac{a^2}{6}
\end{aligned}$$

Now solving for E, we have

$$E(r) = \frac{a^2\rho_0}{6r\epsilon_0}, \quad a < r < b$$

Since the electric field inside a conductor is always zero, $E(r) = 0$ for $b < r < c$. Now for $r > c$, we need to find the electric field due to the conducting cylinder. By Gauss's Law, we have

$$\begin{aligned}
\int \mathbf{E}\cdot d\mathbf{A} &= \frac{Q}{\epsilon_0} \\
2\pi r l E &= \frac{\lambda l}{\epsilon_0}
\end{aligned}$$

This gives $E = \lambda/(2\pi\epsilon_0 r)$. Now adding this with the result for the electric field outside the insulating cylinder, we have

$$E(r) = \frac{a^2\rho_0}{6r\epsilon_0} + \frac{\lambda}{2\pi\epsilon_0 r}, \quad r > c$$

All together, the result is

$$E(r) = \begin{cases} \frac{\rho_0}{\epsilon_0}\left(\frac{r}{2}-\frac{r^2}{3a}\right), & 0 < r < a \\ \frac{a^2\rho_0}{6r\epsilon_0}, & a < r < b \\ 0, & b < r < c \\ \frac{a^2\rho_0}{6r\epsilon_0} + \frac{\lambda}{2\pi r}, & r > c \end{cases}$$

(b) Since we know that the electric field inside a conductor is zero, we know that the linear surface charge density on the inner surface of the conductor must exactly cancel the charge on the insulating cylinder. This gives us

$$\int_0^a \rho(s)(2\pi s l)ds + \sigma_b l = 0$$

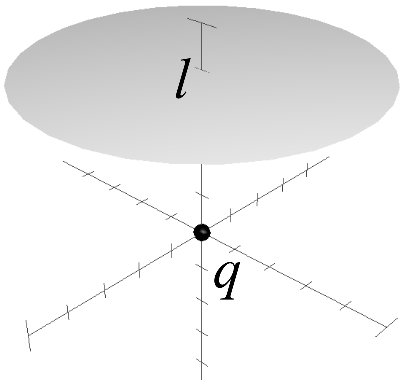
Solving for σ_b and making use of the integrals calculated in part (a), we have

$$\sigma_b = -\frac{2\pi a^2\rho_0}{6}$$

(c) Then since $\sigma_b + \sigma_c = \lambda$, we have

$$\sigma_c = \lambda + \frac{2\pi a^2\rho_0}{6}$$

(∗∗) 6. A point charge q is at the origin. Consider a circular surface of radius a that is normal to $\mathbf{k}$, at a distance l from q. The centre of the circular surface is directly above the origin. See the figure. What is the electric flux through the surface?



Solution: First, note that this is an *open* surface. This means that Gauss's law does not help us in this situation.
Consider an annulus in the circle of radius differential dr. Let $r^2 = x^2 + y^2$. Thus, the magnitude of the electric field on this annulus is

$$E = \frac{q}{4\pi\epsilon_0(r^2 + l^2)}$$

The flux through the surface is given by

$$\Phi = \int \mathbf{E} \cdot d\mathbf{A} = \int E \cos\theta dA$$

Since we're looking at small annuli, we have $dA = 2\pi r dr$. Since the normal vector to the surface points directly upward, we have $\cos\theta = l/\sqrt{r^2 + l^2}$. Thus we have

$$\begin{aligned} \Phi &= \int_0^a E \cos\theta dA \\ &= \int_0^a \left(\frac{q}{4\pi\epsilon_0(r^2 + l^2)}\right)\left(\frac{l}{\sqrt{r^2 + l^2}}\right) 2\pi r dr \\ &= \frac{q}{2\epsilon_0}\left(1 - \frac{l}{\sqrt{a^2 + l^2}}\right) \end{aligned}$$

So the flux through the circular surface is

$$\Phi = \frac{q}{2\epsilon_0}\left(1 - \frac{l}{\sqrt{a^2 + l^2}}\right)$$

(∗ ∗ ∗) 7. Imagine that the Earth were of uniform density and that a tunnel was drilled along the diameter from the North Pole to the South Pole. Assume the Earth is a perfect sphere, and let R and M be the radius and mass of the Earth, respectively. See the figure below. Also shown in the figure is a spherical Gaussian surface of radius r

(a) If an object were dropped into the tunnel, show that it will undergo simple harmonic motion.

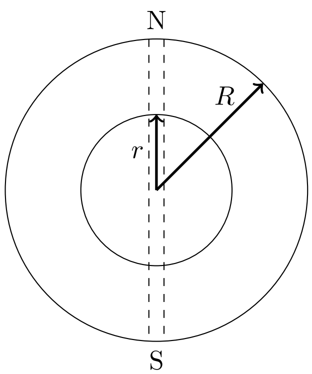
(b) Find its period P of oscillation.

(c) Show that the period P of oscillation is equal to the period of a satellite orbiting Earth just at the surface.

Hint. *Gauss's Law for gravitational fields is*

$$\oint \boldsymbol{g} \cdot d\boldsymbol{A} = -4\pi G \sum_i M_i$$

It has conceptually identical to Gauss's Law for electric fields. The analogy is $\sum Q \to \sum M$, $1/\epsilon_0 \to -4\pi G$, *and* $\boldsymbol{E} \to \boldsymbol{g}$. *Note that* $\boldsymbol{g}$ *is the gravitational field (which is analogous to the electric field* $\boldsymbol{E}$*).*



Solution:

(a) Let the mass density of the Earth be ρ. Imagine a spherical Gaussian surface of radius $r < R$. Then

$$\begin{aligned}
\oint \mathbf{g} \cdot d\mathbf{A} &= -4\pi G \left(\frac{4\pi}{3}\rho r^3\right) \\
\oint g dA &= \\
\oint |\mathbf{g}||d\mathbf{A}| \cos(0) &= \\
g \oint dA &= \\
g(4\pi r^2) &= -4\pi G \left(\frac{4\pi}{3}\rho r^3\right)
\end{aligned}$$

Thus dividing through by $4\pi r^2$,

$$g(r) = -\frac{4}{3}\pi G \rho r$$

From Newton's Second Law, we obtain

$$F = ma = mr''(t) = mg = m\left(-\frac{4}{3}\pi G\rho r(t)\right)$$

Which gives the differential equation

$$r'' + \frac{4\pi G\rho}{3} r = 0$$

This is the differential equation of a simple harmonic oscillator. Thus the object will undergo simple harmonic motion.

(b) The solution to the equation is $r(t) = A\cos(\omega t)$, where $\omega^2 = \dfrac{4\pi G\rho}{3}$. Thus,

$$\begin{aligned} P &= \frac{2\pi}{\omega} \\ &= \frac{2\pi}{\sqrt{\frac{4\pi G\rho}{3}}} \\ &= \sqrt{\frac{3\pi}{G\rho}} \\ &= \sqrt{\frac{3\pi}{G(M/V)}} \\ &= \sqrt{\frac{3\pi V}{GM}} \\ &= \sqrt{\frac{3\pi\frac{4}{3}\pi R^3}{GM}} \\ &= \sqrt{\frac{4\pi^2 R^3}{GM}} \\ &= 2\pi R\sqrt{\frac{R}{GM}} \end{aligned}$$

(c) The period of a satellite orbiting Earth at the surface can be found using the centripetal acceleration formula.

$$|\mathbf{F}| = ma = \frac{mv^2}{R} = \frac{mMG}{R^2}$$

$$v = \sqrt{\frac{MG}{R}} = \frac{2\pi R}{P}$$

$$P = 2\pi R\sqrt{\frac{R}{GM}}$$

We see that the results from parts (b) and (c) match.

(∗ ∗ ∗) 8. Suppose that you have a series RLC circuit in an FM radio. You tune in on a broadcast using a variable capacitor. There are two radio stations broadcasting: Station 1 broadcasts at $\omega_1 = 6 \cdot 10^8$, and Station 2 broadcasts at $\omega_2 = 5.99 \cdot 10^8$. The inductance of the inductor is $L = 10^{-6}$ H, and both stations drive the circuit with the same max voltage.

(a) Find the value of the capacitor that you need in order to tune in to Station 1.

(b) Fix the value of C to be that found in part (a). If the mean power consumed by the circuit when listening to Station 1 (in the absence of station 2) is 100 times the mean power consumed by the circuit when listening to Station 2 (in the absence of station 1), what is the value of the resistor R?

Solution:

(a) Since it is tuned in to ω_1, that means the circuit must resonate at this frequency. So $\omega_1 = \omega_0 = 1/\sqrt{LC}$. Thus $C = 2.778$ pF.

(b) $P = I_{\text{rms}} V_{\text{rms}} \cos\phi$, and $P_1/P_2 = 100$, if P_1 is the average power dissipated by the circuit when listening to station 1, and P_2 is the average power dissipated by the circuit when listening to station 2. Using this, we have

$$\frac{I_{\text{rms1}} V_{\text{rms1}} \cos\phi_1}{I_{\text{rms2}} V_{\text{rms2}} \cos\phi_2} = 100$$

When tuned to frequency 1, the circuit is at resonance, so $\phi_1 = 0$. We are also told that the input voltages are the same at each frequency, so the rms voltages cancel out. Thus we have

$$\frac{I_{\text{rms1}}}{I_{\text{rms2}} \cos\phi_2} = 100$$

Since $I_{\text{rms}} = I_0/\sqrt{2}$, we have $I_{\text{rms1}} = I_0/\sqrt{2} = V_0/\sqrt{2}Z = V_0/\sqrt{2}R$ ($Z = R$ for station 1 because it is at resonance). We also have

$$\begin{aligned} I_{\text{rms2}} &= V_0/\sqrt{2}Z \\ &= \frac{V_0}{\sqrt{2}\sqrt{(X_L - X_C)^2 + R^2}} \\ &= \frac{V_0}{\sqrt{2}\sqrt{(\omega_2 L - 1/\omega_2 C)^2 + R^2}} \end{aligned}$$

Thus we have

$$\frac{V_0}{\sqrt{2}R} = \frac{100 V_0 \cos\phi_2}{\sqrt{2}\sqrt{(\omega_2 L - 1/\omega_2 C)^2 + R^2}}$$

Since

$$\tan\phi_2 = \frac{\omega_2 L - 1/\omega_2 C}{R}$$

we have

$$\cos\phi_2 = \frac{R}{\sqrt{(\omega_2 L - 1/\omega_2 C)^2 + R^2}}$$

This produces

$$\frac{1}{R} = \frac{100R}{(\omega_2 L - 1/\omega_2 C)^2 + R^2}$$

So then

$$100R^2 = (\omega_2 L - 1/\omega_2 C)^2 + R^2$$

Which yields

$$R = \frac{|\omega_2 L - 1/\omega_2 C|}{\sqrt{99}} = \frac{|599 - 601|}{\sqrt{99}} = 0.201\ \Omega$$

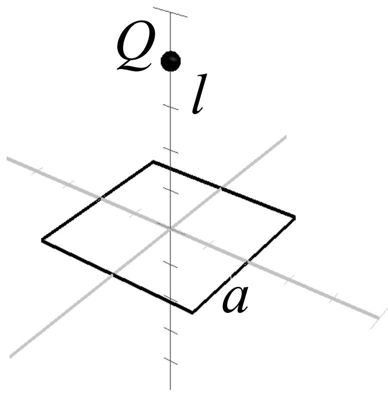
(∗ ∗ ∗) 9. A charged square insulating wire of side length a, of uniform linear charge density λ, is centred about the origin of the xy plane with sides parallel to the x and y axes. A charge Q lies a distance l above its centre. See the figure.

(a) Find the magnitude and direction of the force $\mathbf{F}$ acting on the charge Q.

(b) Find the approximate magnitude of the force $\mathbf{F}$ acting on the charge Q for $l \gg a$.

(c) Find the approximate magnitude of the force $\mathbf{F}$ acting on the charge Q for $l \ll a$.

(d) If the charge Q has a mass m, and $\lambda < 0$, and $Q > 0$, find the frequency of small oscillations of Q about the origin.



Hint: The following integral may be useful:

$$\int \frac{du}{(u^2+\alpha^2)^{3/2}} = \frac{u}{\alpha^2\sqrt{u^2+\alpha^2}}$$

Solution:

(a) Let λ be the linear charge density of the square insulator. Then, a little element of charge on the insulator is given by $dq = \lambda dx$, and the distance from any point (x, y) on the square to the charge is given by $r = \sqrt{x^2+y^2+l^2}$. First, note that the x and y components of the force on Q will cancel out, and it will only be pushed upwards. Thus, we only need to find the z component of force on Q from one of the sides, then multiply by 4. Let's integrate along the x direction.

$$\begin{aligned} dF_z &= \frac{Qdq\cos\theta}{4\pi\epsilon_0 r^2} \\ &= \frac{Q\lambda dx\cos\theta}{4\pi\epsilon_0 r^2} \\ &= \frac{Q\lambda dx}{4\pi\epsilon_0 r^2}\cdot\frac{l}{r} \\ &= \frac{Ql\lambda dx}{4\pi\epsilon_0(x^2+y^2+l^2)^{3/2}} \end{aligned}$$

Thus the force on Q is given by $\mathbf{F} = F_z\mathbf{k}$. Note that we set $y = a/2$ because we integrate along one side of the square, then multiply the final result by 4.

$$\begin{aligned} F_z &= \int_{-a/2}^{a/2} \frac{Ql\lambda dx}{4\pi\epsilon_0(x^2+y^2+l^2)^{3/2}} \\ &= \frac{Ql\lambda}{4\pi\epsilon_0}\cdot\int_{-a/2}^{a/2}\frac{dx}{(x^2+(a/2)^2+l^2)^{3/2}} \\ &= \frac{Ql\lambda}{\pi\epsilon_0}\cdot\int_{-a/2}^{a/2}\frac{2dx}{(4x^2+(4l^2+a^2))^{3/2}} \end{aligned}$$

Evaluating the integral with the help of the hint (I show the steps for evaluating the integral below if you're interested) gives:

$$\frac{Ql\lambda}{\pi\epsilon_0} \cdot \int_{-a/2}^{a/2} \frac{2dx}{(4x^2 + (4l^2 + a^2))^{3/2}} = \frac{Ql\lambda}{\pi\epsilon_0(4l^2 + a^2)} \cdot \left(\frac{2x}{\sqrt{4x^2 + 4l^2 + a^2}} \right)\Bigg|_{-a/2}^{a/2}$$

$$= \frac{Qla\sqrt{2}\lambda}{\pi\epsilon_0(4l^2 + a^2)\sqrt{a^2 + 2l^2}}$$

Thus the force on the charge (multiplying by 4) Q is

$$\mathbf{F} = \frac{Qla\lambda 4\sqrt{2}}{\pi\epsilon_0(4l^2 + a^2)\sqrt{a^2 + 2l^2}}\mathbf{k}$$

(b) If $l \gg a$, then $4l^2 + a^2 \approx 4l^2$, and $\sqrt{a^2 + 2l^2} \approx \sqrt{2}l$. Then we have

$$\mathbf{F} \approx \frac{Qa\lambda}{\pi\epsilon_0 l^2}\mathbf{k}$$

which agrees with the model of approximating the charged square as a point charge of magnitude $4a\lambda$, at distance l from the charge Q.

(c) If $l \ll a$, then we can approximate $4l^2 + a^2 \approx a^2$, and $\sqrt{a^2 + 2l^2} \approx a$. Then we have

$$\mathbf{F} \approx \frac{\lambda Q l 4\sqrt{2}}{\pi\epsilon_0 a^2}$$

(d) Relabelling $l = z$, we can apply Newton's second law to obtain

$$mz''(t) = \frac{Q\lambda l 4\sqrt{2}}{\pi\epsilon_0 a^2} z(t)$$

The angular frequency is then (we have a minus sign because $Ql < 0$)

$$\omega = \sqrt{-\frac{Q\lambda l 4\sqrt{2}}{m\pi\epsilon_0 a^2}}$$

which gives the frequency

$$f = \frac{1}{2\pi}\sqrt{-\frac{Q\lambda l 4\sqrt{2}}{m\pi\epsilon_0 a^2}}$$

Evaluation of the Integral (For those interested students)

$$\frac{Ql\lambda}{\pi\epsilon_0} \cdot \int_{-a/2}^{a/2} \frac{2dx}{(4x^2 + (4l^2 + a^2))^{3/2}}$$

First, we find the antiderivative.
Let $2x = \sqrt{4l^2 + a^2}\tan\theta \Rightarrow 2dx = \sqrt{4l^2 + a^2}\sec^2\theta d\theta$. Then

$$\int \frac{2dx}{(4x^2 + (4l^2 + a^2))^{3/2}} = \int \frac{\sqrt{4l^2 + a^2}\sec^2\theta d\theta}{[(4l^2 + a^2)\tan^2\theta + (4l^2 + a^2)]^{3/2}}$$

$$= \frac{1}{4l^2 + a^2} \cdot \int \cos\theta d\theta = \frac{1}{4l^2 + a^2}\sin\theta$$

Expressing $\sin\theta$ in terms of x gives

$$\frac{1}{4l^2+a^2}\sin\theta = \frac{1}{4l^2+a^2}\cdot\frac{2x}{\sqrt{4x^2+4l^2+a^2}}$$

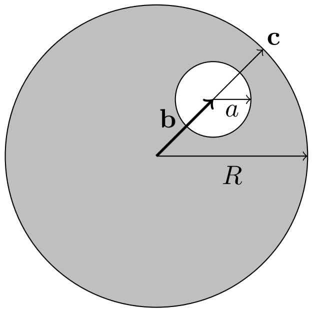
Now, plugging this back into the original expression gives:

$$\begin{aligned}\frac{Ql\lambda}{\pi\epsilon_0}\cdot\int_{-a/2}^{a/2}\frac{2dx}{(4x^2+(4l^2+a^2))^{3/2}} &= \frac{Ql\lambda}{\pi\epsilon_0(4l^2+a^2)}\cdot\left(\frac{2x}{\sqrt{4x^2+4l^2+a^2}}\right)\Bigg|_{-a/2}^{a/2} \\ &= \frac{Qla\sqrt{2}\lambda}{\pi\epsilon_0(4l^2+a^2)\sqrt{a^2+2l^2}}\end{aligned}$$

(∗ ∗ ∗) 10. Consider an insulating sphere of radius R, centred at the origin with uniform charge density ρ. A spherical cavity of radius a is scooped out, with centre at $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, where $a + |\mathbf{b}| < R$. See the figure below.

(a) Find the magnitude and direction of the electric field at any point within the cavity.

(b) Find the magnitude and direction of the electric field at the point $\mathbf{c} = R\hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is the unit vector in the $\mathbf{b}$ direction.



Solution:

(a) This problem requires the application of superposition. First, suppose that the sphere is completely solid. Then, we can use Gauss's law to find the electric field at any distance r from its centre. First, take the origin of the coordinate system to be at the centre of the sphere. Then we have

$$\begin{aligned} \oint \mathbf{E}_1 \cdot d\mathbf{A} &= \frac{\sum Q}{\epsilon_0} \\ &= \frac{4\pi r^3 \rho}{3\epsilon_0} \\ &= |\mathbf{E}_1|(4\pi r^2) \end{aligned}$$

Thus

$$|\mathbf{E}_1| = \frac{\rho r}{3\epsilon_0}$$

So then

$$\mathbf{E}_1 = \frac{\rho r}{3\epsilon_0}\hat{\mathbf{r}} = \frac{\rho}{3\epsilon_0}\langle x, y, z \rangle$$

We now must consider the cavity. This can be achieved by placing a sphere of charge density $-\rho$ centred at $\mathbf{b}$, with radius a. The magnitude of the electric field due to this second sphere will be $|\mathbf{E}_2| = \rho r'/3\epsilon_0$, where r' is the distance from $\mathbf{b}$. Note that this formula only holds for $r' < a$. Thus we have

$$\mathbf{E}_2 = \frac{\rho}{3\epsilon_0}\langle b_1 - x, b_2 - y, b_3 - z \rangle$$

The electric field within the cavity is then

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 \\ &= \frac{\rho}{3\epsilon_0}\mathbf{b} \end{aligned}$$

Note that this electric field (within the cavity) is constant.

(b) The formula that we obtained for the electric field $\mathbf{E}_1$ still holds, because $\mathbf{c}$ lies at the surface of the sphere. However, we must obtain a new formula for the electric field due to the cavity. The electric field due to the negative mini-sphere will be

$$\begin{aligned} |\mathbf{E}_2| &= \frac{-\rho(4\pi a^3/3)}{4\pi\epsilon_0 r'^2} \\ &= \frac{-\rho a^3}{3\epsilon_0 r'^2} \end{aligned}$$

Since we are looking at the surface of the sphere, $r' = R - |\mathbf{b}|$. The direction of the field will be in the direction of $\mathbf{b}$. Thus,

$$\begin{aligned} \mathbf{E}_2 &= \frac{-\rho a^3}{3\epsilon_0 (R - |\mathbf{b}|)^2}\hat{\mathbf{b}} \\ &= \frac{-\rho a^3}{3\epsilon_0 (R - |\mathbf{b}|)^2 |\mathbf{b}|}\mathbf{b} \end{aligned}$$

The electric field at this point $\mathbf{c}$ is then

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 \\ &= \frac{\rho \mathbf{b}}{3\epsilon_0}\left(R - \frac{a^3}{|\mathbf{b}|(R - |\mathbf{b}|)^2}\right) \end{aligned}$$

Waves:

$$v = \sqrt{\frac{T}{\mu}},\ k = \frac{2\pi}{\lambda},\ P = \tfrac{1}{2}\mu\omega^2 A^2 v,\ p_o = \rho\omega v s_0$$

$$v = \sqrt{\frac{\gamma RT}{M}}, \qquad I = \frac{P_{\text{av}}}{4\pi r^2}, \qquad \beta = 10dB\log_{10}\left(\tfrac{I}{I_0}\right), \qquad \text{Doppler Effect } f' = f_0\left(\frac{v \pm v_L}{v \mp v_S}\right)$$

$$\text{Beats: } \Delta f = f_2 - f_1, \qquad y = A\cos(kx \mp \omega t + \phi)$$

$$\text{Interference: } k\Delta x + \Delta\phi = 2\pi n \text{ or } \pi(2n+1),\ n = 0, \pm1, \pm2, \pm3, \pm4, \ldots$$

$$\text{Standing Waves } f_m = \frac{mv}{2L},\ m = 1, 2, 3, \ldots,\ f_m = \frac{mv}{4L},\ m = 1, 3, 5, \ldots$$

Constants:

$$k = \frac{1}{4\pi\epsilon_0} \approx 9 \times 10^9\,\text{Nm}^2/\text{C}^2, \qquad \epsilon_0 = 8.84 \times 10^{-12}\,\text{C}^2/\text{Nm}^2, \qquad e = 1.6 \times 10^{-19}\,\text{C}$$

$$\mu_0 = 4\pi \times 10^{-7}\,\text{Tm/A}, \qquad c = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 299,792,458\,\text{m/s}$$

Point Charge:

$$|\mathbf{F}| = \frac{k|q_1 q_2|}{r^2},\ |\mathbf{E}| = \frac{k|q|}{r^2},\ V = \frac{kq}{r} + \text{Constant}$$

Electric potential and potential energy $\Delta V = V_a - V_b = \int_a^b \mathbf{E}\cdot d\mathbf{l} = -\int_b^a \mathbf{E}\cdot d\mathbf{l}$

$$E_x = -\frac{dV}{dx}, \qquad \mathbf{E} = -\nabla V, \qquad \Delta U = U_a - U_b = q(V_a - V_b)$$

Maxwell's Equations:

$$\int_S \mathbf{E}\cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\epsilon_0} = 4\pi k Q_{\text{enc}} \qquad \int_S \mathbf{B}\cdot d\mathbf{A} = 0$$

$$\int_C \mathbf{B}\cdot d\mathbf{l} = \mu_0\left(I_{\text{enclosed}}\right) + \epsilon_0\mu_0\frac{d\Phi_E}{dt} \qquad \int_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi_B}{dt}$$

Where S is a closed surface and C is a closed curve. $\Phi_E = \int \mathbf{E}\cdot d\mathbf{A}$ and $\Phi_B = \int \mathbf{B}\cdot d\mathbf{A}$

Energy Density:

$$u_E = \frac{1}{2}\epsilon_0 E^2 \text{ and } u_B = \frac{1}{2\mu_0}B^2 \text{ (energy per volume)}$$

Forces:

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B},\ \mathbf{F} = I\mathbf{L}\times\mathbf{B}$$

Capacitors:

$q = CV$, $U_C = \frac{1}{2}\cdot\frac{q^2}{C}$, For parallel plate capacitor with vacuum (air): $C = \frac{\epsilon_0 A}{d}$, $C_{\text{dielectric}} = KC_{\text{vacuum}}$

Inductors:

$\mathcal{E}_L = -L\frac{dI}{dt}$, $U_L = \frac{1}{2}LI^2$, where $L = N\Phi_B/I$ and N is the number of turns.

For a solenoid $B = \mu_0 nI$ where n is the number of turns per unit length.

DC Circuits: $V_R = IR$, $P = VI$, $P = I^2R$

(For RC circuits)$q = ae^{-t/\tau} + b, \tau = RC,$ a and b are constants

(For LR circuits)$I = ae^{-t/\tau} + b, \tau = L/R,$ a and b are constants

AC circuits: $X_L = \omega L$, $X_C = 1/(\omega C)$, $V_C = X_C I$, $V_L = X_L I$

$$V = ZI,\ Z = \sqrt{(X_L - X_C)^2 + R^2},\ P_{\text{average}} = I_{\text{rms}}^2 R,\ I_{\text{rms}} = \frac{I_{\text{max}}}{\sqrt{2}}$$

If $V = V_0\cos(\omega t)$, then $I = I_{\text{max}}\cos(\omega t - \phi)$, where $\tan\phi = \frac{X_L - X_C}{R}$, $P_{\text{av}} = V_{\text{rms}} I_{\text{rms}}\cos\phi$

Additional Equations: $d\mathbf{B} = \frac{\mu_0}{4\pi}\cdot\frac{Id\mathbf{l}\times\mathbf{r}}{r^3}$

LRC Oscillations: $q = A_0 e^{-\frac{Rt}{2L}}\cos(\omega t + \phi)$, where $\omega = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}$ and $\omega_0^2 = \frac{1}{LC}$