EUS Tutoring Session Review Problem Set **SOLUTIONS**

Mathematics 255 - Midterm 1

Ordinary Differential Equations

Note on notation: Whenever $\log(x)$ is used without a subscript to indicate the base, it is assumed to be base e in math courses. Thus in this review package, $\log(x)$ and $\ln(x)$ are used interchangeably. For inverse trigonometric functions, $\sin^{-1}(x) = \arcsin(x)$, and the other inverse trigonometric functions are similarly denoted.

The solutions to these problems will be posted on ubcengineers.ca → Services → Academic Services → Tutoring. If you believe that there is an error in an answer key, or if you have suggestions for improvement of EUS tutoring sessions, please e-mail us at: tutoring@ubcengineers.ca.

The contents of this package include: Slope Fields, Separable Equations, First Order Linear Equations, Exact Equations, Euler's Method, Autonomous Equations, Existence and Uniqueness Theorem

1) Solve the following differential equation for $y = y(x)$. $\frac{dy}{dx} - \frac{2xy}{x^2+1} = 1$

Solution

Find the integrating factor $\mu(x) = e^{\int \frac{-2x}{x^2+1}dx} = e^{-\log(x^2+1)} = \frac{1}{x^2+1}$.

Multiply both sides of the equation by the integrating factor → $\frac{1}{x^2+1}\frac{dy}{dx} - \frac{2x}{(x^2+1)^2}y = \frac{1}{x^2+1} = \left(\frac{y}{1+x^2}\right)'$

Integrating, $\frac{y}{x^2+1} = \arctan x + C \Rightarrow y = (x^2+1)\arctan x + C(x^2+1)$

2) Solve the following differential equation for $r = r(\theta)$. $\tan\theta \frac{dr}{d\theta} - r = \tan^2\theta$

Solution

Put this in the form in which we will use the integrating factor $\mu(\theta)$.
$r' - r \cdot \cot\theta = \tan\theta$.

$\mu = e^{\int -\cot\theta d\theta} = e^{-\log\sin x} = \csc\theta$

Differential equation will become $r'\csc\theta - r\csc\theta\cot\theta = \sec\theta = (r\csc\theta)'$

Integrating, $\log(\sec\theta + \tan\theta) + C = r\csc\theta$

Thus, $r(\theta) = C\sin\theta + \sin\theta\log(\sec\theta + \tan\theta)$

3) Solve the following differential equation explicitly as a function of x. $(y^2+1)dx-(x^2+1)dy=0$

Solution

Separate variables: $\dfrac{dy}{y^2+1}=\dfrac{dx}{x^2+1}$

Integrate: $\arctan y=\arctan x+C \Rightarrow y=\tan(\arctan x+C)=\dfrac{x+C'}{1-C'x}$.

4) Solve the following initial value problem. $ydy+xdx=3xy^2dx$, $y(2)=1$

Solution

We first move all of the differentials to one side to obtain: $(x-3xy^2)dx+ydy=0$.

We check for exactness: $\dfrac{\partial(x-3xy^2)}{\partial y}=-6xy\neq 0=\dfrac{\partial y}{\partial x}$. The equation is *not* exact, so we try an integrating factor $\mu=\mu(x)$.

We obtain $\dfrac{\partial(\mu(x)\cdot(x-3xy^2))}{\partial y}=\dfrac{\partial(\mu(x)\cdot y)}{\partial x}\Rightarrow -\mu 6x=\mu'\Rightarrow \mu(x)=e^{-3x^2}$

Now we have the differential equation $e^{-3x^2}(x-3xy^2)dx+e^{-3x^2}ydy=0$.

Integrating, $\displaystyle\int e^{-3x^2}(x-3xy^2)dx=(1-3y^2)\int xe^{-3x^2}dx=(3y^2-1)e^{-3x^2}/6$

Thus, the solution to the differential equation is $\Phi(x,y)=(3y^2-1)e^{-3x^2}/6=C$

5) Solve the following differential equation. $x\log x dy+\sqrt{1+y^2}dx=0$

Solution

The equation is separable, so we rearrange to obtain: $\dfrac{-dy}{\sqrt{1+y^2}}=\dfrac{dx}{x\log x}$.

Integrating, $\log(\sqrt{1+y^2}-y)=\log\log x+C$.

6) Solve the following differential equation. $e^{x+1}\tan y dx+\cos y dy=0$

Solution

Separate variables: $e^{x+1}dx=-\cot y\cos y dy=-\dfrac{\cos^2 y}{\sin y}dy=-\dfrac{1-\sin^2 y}{\sin y}dy=(\sin y-\csc y)dy$

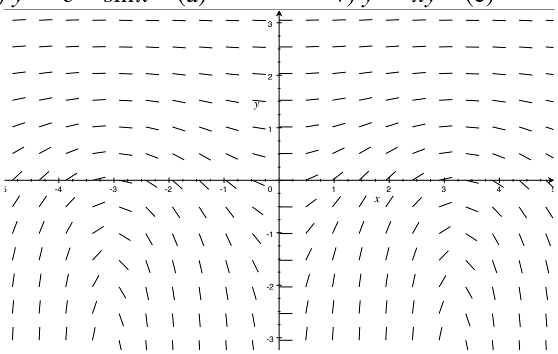
Integrating, $-\cos y+\log|\csc y+\tan y|=e^{x+1}+C$

7) Match the following differential equations with their slope fields.

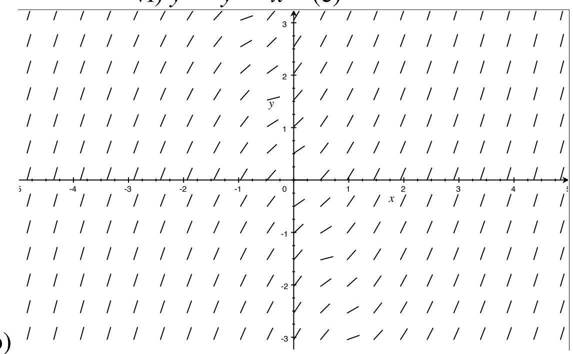
Solution

i) $y' = 4x - 2y/x$ = (d) ii) $y' = \sqrt{|3x+y|}$ = (b) iii) $y' = 1 + 2xy$ = (f)

iv) $y' = e^{-y}\sin x$ = (a) v) $y' = xy$ = (c) vi) $y' = y^2 - x$ = (e)



a)



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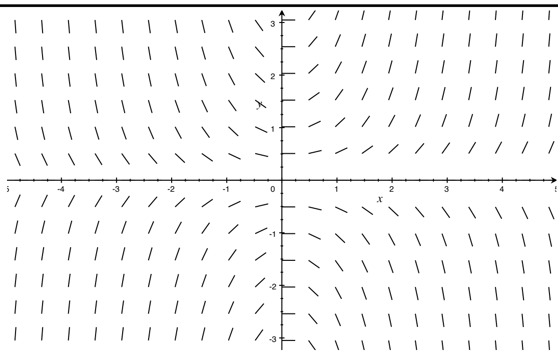

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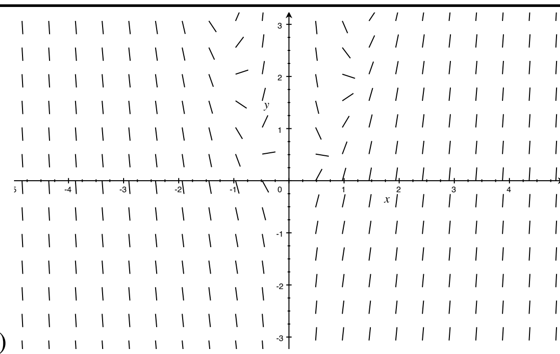

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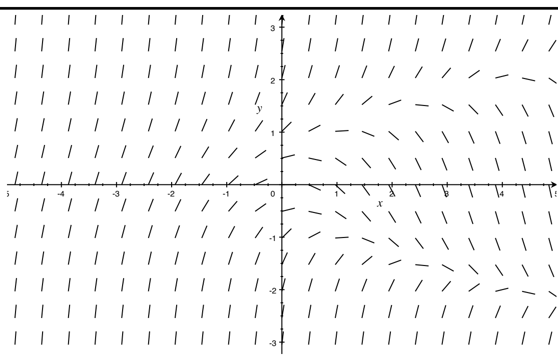

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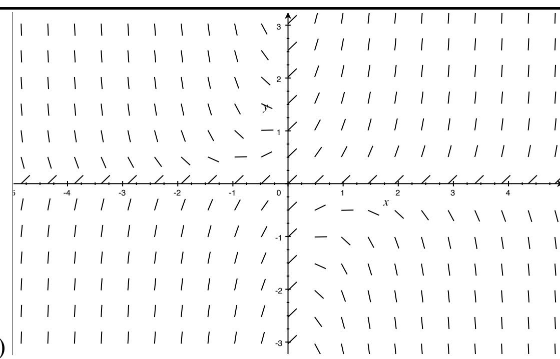


f)

8) Solve the equation. $\frac{dr}{d\theta} = (r + e^{-\theta})\tan\theta$

Solution

We will rearrange this into the first order linear form: $r' - r\tan\theta = e^{-\theta}\tan\theta$

The integrating factor will be $\mu(x) = e^{\int -\tan\theta d\theta} = e^{\log\cos\theta} = \cos\theta$

Multiplying through by the integrating factor, $r'\cos\theta - r\sin\theta = e^{-\theta}\sin\theta = (r\cos\theta)'$

Integrating, $r\cos\theta = \int e^{-\theta}\sin\theta d\theta = \frac{-e^{-\theta}}{2}(\cos\theta + \sin\theta) + C$

$r(\theta) = \frac{-e^{-\theta}}{2}(1 + \tan\theta) + C\sec\theta$

9) Solve the equation $\tan\theta\frac{dr}{d\theta} - r = \tan^2\theta$

Solution

Put this in the form in which we will use the integrating factor $\mu(\theta)$.

$r' - r\cdot\cot\theta = \tan\theta.$

$\mu = e^{\int -\cot\theta d\theta} = e^{-\log\sin x} = \csc\theta$

Differential equation will become $r'\csc\theta - r\csc\theta\cot\theta = \sec\theta = (r\csc\theta)'$

Integrating, $\log(\sec\theta + \tan\theta) + C = r\csc\theta$

Thus, $r(\theta) = C\sin\theta + \sin\theta\log(\sec\theta + \tan\theta)$

10) Solve the equation. $xy' - y = x^2\sin x$

Solution

Rearranging into standard form, $y' - \frac{y}{x} = x\sin x$

The integrating factor is $\mu(x) = e^{\int -1/x dx} = e^{-\log x} = \frac{1}{x}$

Multiplying through, $\frac{y'}{x} - \frac{y}{x^2} = \sin x = \left(\frac{y}{x}\right)'$

Integrating, $\left(\frac{y}{x}\right) = -\cos x + C$

$y(x) = -x\cos x + Cx.$

11) Solve the equation. $x\dfrac{dy}{dx}+2y=\dfrac{-\sin x}{x}$

Solution

Rearange the equation into standard form: $\dfrac{dy}{dx}+\dfrac{2}{x}y=\dfrac{-\sin x}{x^2}$

Find the integrating factor: $\mu(x)=e^{\int \frac{2}{x}dx}=e^{\log x^2}=x^2$

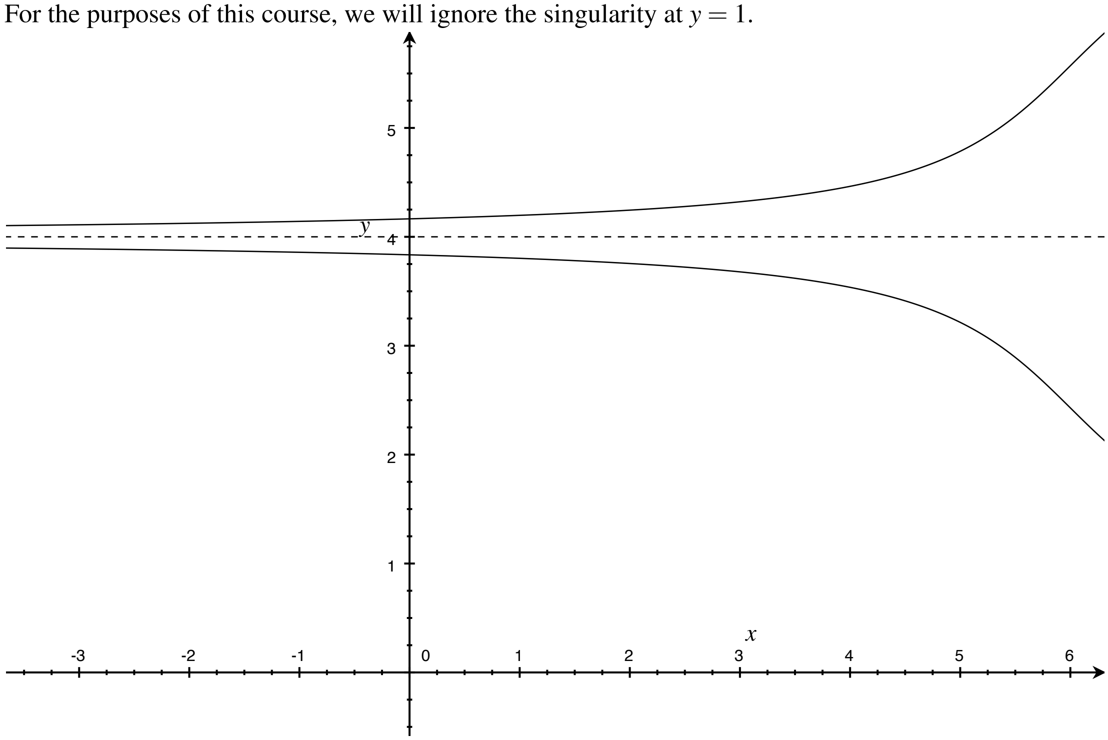
Multiplying through, $x^2\dfrac{dy}{dx}+2xy=-\sin x=(x^2y)'$

Integrating, $(yx^2)=\cos x+C \Rightarrow y=x^{-2}\cos x+Cx^{-2}$

12) Create a phase diagram for the following autonomous equation. $y'=\dfrac{y-4}{y-1}$

Solution

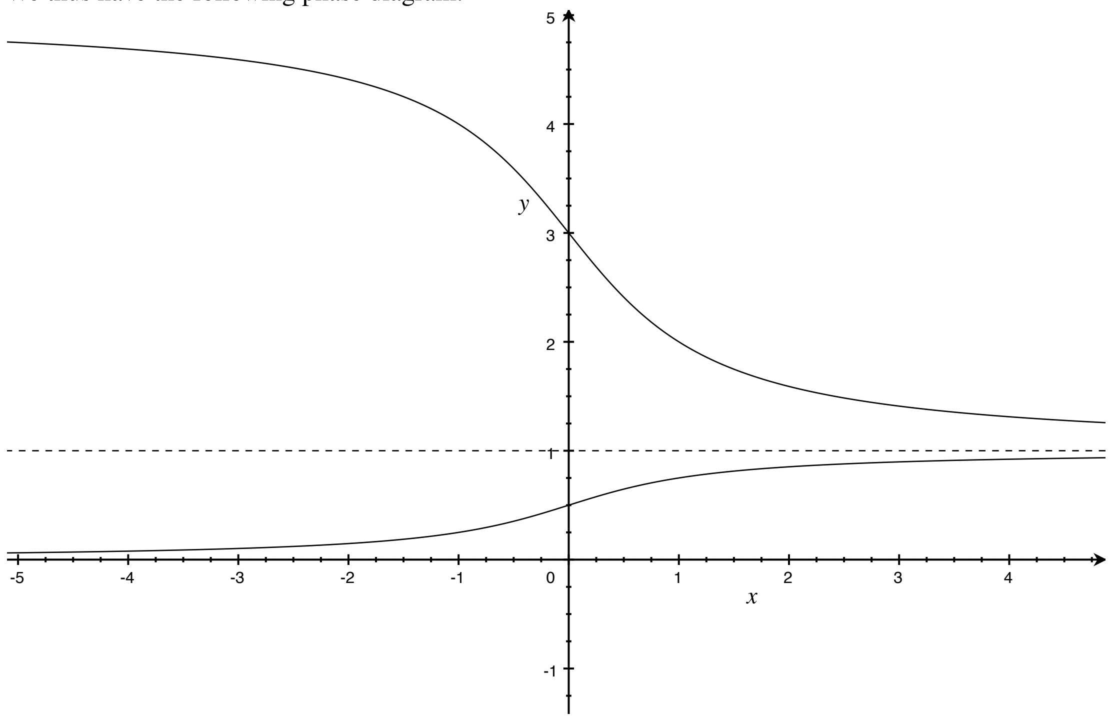
For the purposes of this course, we will ignore the singularity at $y=1$.



We have a source at $y=4$.

13) Create a phase diagram for the following autonomous equation. $y' = \sqrt{y} - y^2$

Solution
The critical points are at $y = 0, 1$.
For $y \in (0,1)$, $y' > 0$
For $y \in (1,\infty)$, $y' < 0$
We thus have the following phase diagram:

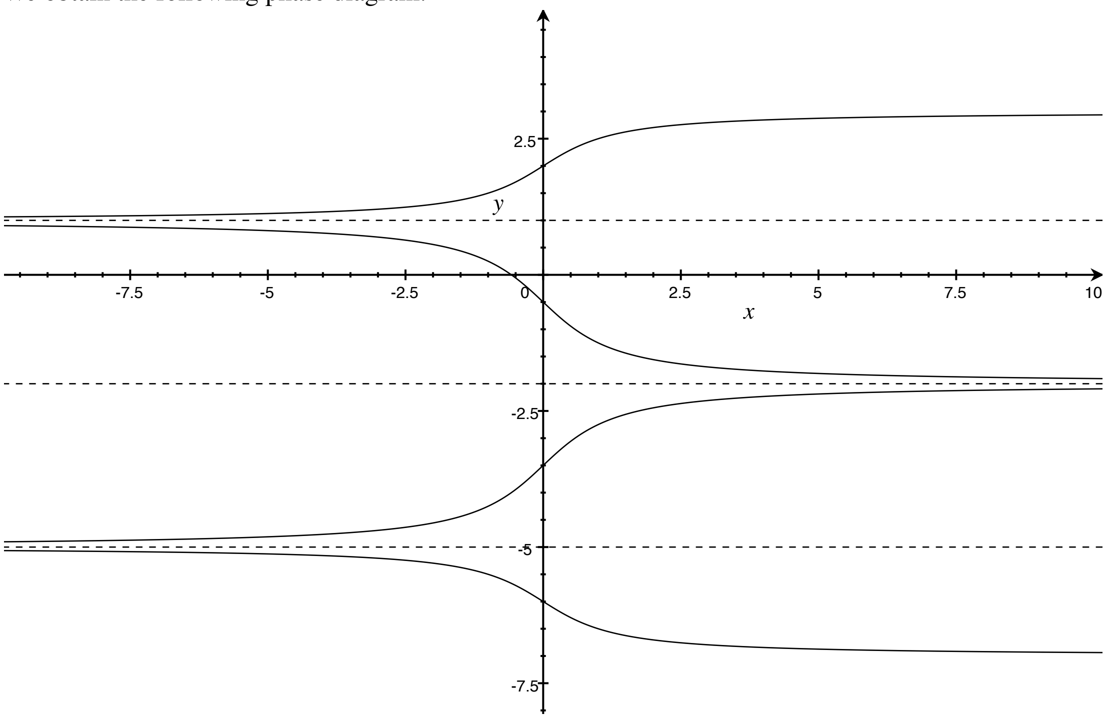


14) Create a phase diagram for the following autonomous equation. $y' = y^3 + 6y^2 + 3y - 10$

Solution

We factor $y' = y^3 + 6y^2 + 3y - 10 = (y+5)(y+2)(y-1)$

For $y \in (-\infty, -5)$, $y' < 0$
For $y \in (-5, -2)$, $y' > 0$
For $y \in (-2, 1)$, $y' < 0$
For $y \in (1, \infty)$, $y' > 0$
We obtain the following phase diagram:



15) Create a phase diagram for the following autonomous equation for $y \in [-2\pi, 2\pi]$. $y' = \sin y \log y$

Solution

Even though the problem says $y \in [-2\pi, 2\pi]$, we must restrict it further because the logarithm only accepts positive values.

First of all, we must see if y' will approach 0 as y approaches 0.

It is left as an exercise (L'Hopital's Rule) to compute $\lim_{y \to 0} \sin y \log y = 0$

We now compute the values of $f(y) = \sin y \log y$ at points between critical points.

The critical points are: $y \in \{1, \pi, 2\pi\}$.

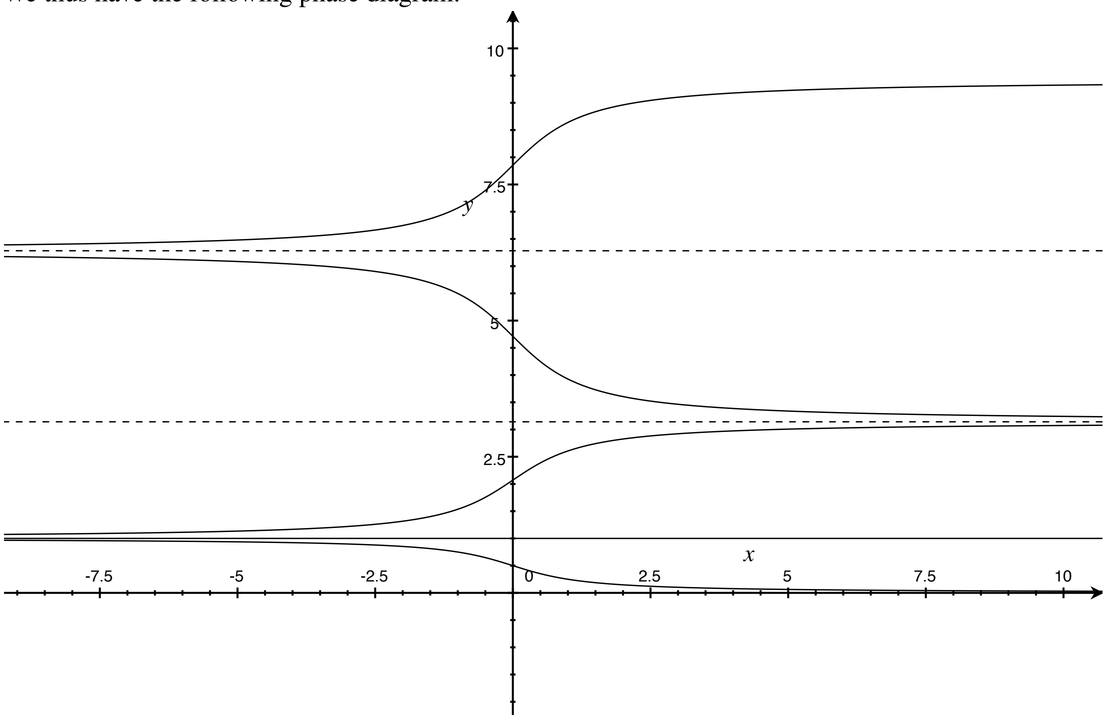
For $y \in (0,1)$, $y' < 0$

For $y \in (1,\pi)$, $y' > 0$

For $y \in (\pi, 2\pi)$, $y' < 0$

For $y \in (2\pi, 3\pi)$, $y' > 0$

We thus have the following phase diagram:



16) Solve the following differential equation

$$x\sqrt{x^2+y^2}dx - \frac{x^2y}{y-\sqrt{x^2+y^2}}dy = 0$$

Solution

This equation is clearly not separable, so we test for exactness.

$$\frac{\partial}{\partial y}\left(x\sqrt{x^2+y^2}\right) = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\frac{\partial}{\partial x}\left(-\frac{x^2y}{y-\sqrt{x^2+y^2}}\right) = \frac{2xy}{\sqrt{x^2+y^2}-y} = \frac{x^3y}{(\sqrt{x^2+y^2}-y)^2\sqrt{x^2+y^2}} = -\frac{xy(2y\sqrt{x^2+y^2}-x^2-2y^2)}{(x^2+y^2-2y\sqrt{x^2+y^2}+y^2)\sqrt{x^2+y^2}}$$

$$\frac{\partial}{\partial x}\left(\frac{x^2y}{\sqrt{x^2+y^2}-y}\right) = \frac{xy}{\sqrt{x^2+y^2}}$$

The equation is exact. We integrate.

$$\Phi(x,y) = \int x\sqrt{x^2+y^2}dx = \frac{1}{3}(x^2+y^2)^{3/2} + f(y)$$

Differentiate with respect to y :

$$y\sqrt{x^2+y^2} + f'(y) = \frac{x^2y}{\sqrt{x^2+y^2}-y}$$

$$y(x^2+y^2) - y^2\sqrt{x^2+y^2} + f'(y)(\sqrt{x^2+y^2}-y) = x^2y$$

$$f'(y) = y^2 \Rightarrow f(y) = y^3/3 + C$$

Multiplying through by 3, we obtain

$$(x^2+y^2)^{3/2} + y^3 = C'$$

17) Solve the following differential equation $(e^x \sin y + e^{-y})dx - (xe^{-y} - e^x \cos y)dy = 0$

Solution

This equation is clearly not separable, thus we must test for exactness.

$$\frac{\partial}{\partial y}\left(e^x \sin y + e^{-y})\right) = e^x \cos y - e^{-y}$$

$$\frac{\partial}{\partial x}\left(-xe^{-y} + e^x \cos y)\right) = -e^{-y} + e^x \cos x$$

The equation is exact, so we integrate.$\Phi(x,y) = \int e^x \sin y + e^{-y} dx = e^x \sin y + xe^{-y} + f(y)$

Differentiate $\Phi(x,y)$ with respect to y: $\frac{\partial}{\partial x}(e^x \sin y + xe^{-y} + f(y)) = e^x \cos y - xe^{-y} + f'(y)$. Thus $f'(y) = 0$.
The solution is

$$\Phi(x,y) = e^x \sin y + xe^{-y} = C$$

18) Determine whether or not the initial value problem $y' = \cos(x+y)$, $y(x_0) = y_0$ has a unique solution defined on all of $\mathbb{R}$.

Solution
Let $f(x,y) = \cos(x+y)$. $f(x,y)$ is continuous everywhere, and $f_y(x,y) = -\sin(x+y)$ is also continuous everywhere. Thus, by the existence and uniqueness theorem, the initial value problem given has a unique solution defined on all of $\mathbb{R}$.

19) Solve the equation $2yx^3 dy + (3x^2y^2 + x^3y^2 + 1)dx = 0$

Solution

The equation is clearly not separable, so we test for exactness.

$$\frac{\partial}{\partial x}(2yx^3) = 6yx^2$$

$$\frac{\partial}{\partial y}(3x^2y^2 + x^3y^2 + 1) = 6x^2y + 2x^3y$$

The equation is not exact, so we will try and integrating factor $\mu = \mu(x)$.

We find $\mu(x) = e^x$.

Thus, we integrate $\Phi(x,y) = \int 2yx^3 e^x dy = y^2x^3e^x + f(x)$

Differentiating with respect to x, $\Phi_x(x,y) = 3x^2y^2e^x + x^3y^2e^x + f'(x) = 3x^2y^2e^x + x^3y^2e^xe^x$. We find $f(x) = x + C$.
Thus the solution is

$$\Phi(x,y) = y^2x^3e^x + e^x = C$$

20) For the given differential equation, use Euler's Method with step size $\frac{1}{2}$ to estimate $y(2)$ if the solution passes through (1,0). $\dfrac{dy}{dx} = x - \dfrac{y^2}{4}$

Solution

Let $f(x,y) = x - \dfrac{y^2}{4}$

$y(3/2) \approx y(1) + f(1, y(1))(1/2) = 1/2$
$y(2) \approx y(3/2) + f(3/2, y(3/2))(1/2) = 39/32$

21) Use Euler's Method with step size 0.2 to estimate $y(1)$, where $y(x)$ is the solution of the initial value problem $y' = xy - x^2$, $y(0) = 1$.

Solution
Let $f(x,y) = xy - x^2$
$y(0.2) \approx y(0) + f(0, y(0))(0.2) = 1$
$y(0.4) \approx y(0.2) + f(0.2, y(0.2))(0.2) = 29/25$
$y(0.6) \approx y(0.4) + f(0.4, y(0.4))(0.2) = 763/625$
$y(0.8) \approx y(0.6) + f(0.6, y(0.6))(0.2) = 1.369792$
$y(1) \approx y(0.8) + f(0.8, y(0.8))(0.2) = 1.461$

22) Show that the function

$$y_1(x) = \begin{cases} 0, & x < 0 \\ x^3, & x \geq 0 \end{cases}$$

is a solution of the initial value problem $xy' = 3y$, $y(0) = 0$. Show that $y_2(x) \equiv 0$ is a second solution. Explain why this does not contradict the existence and uniqueness theorem.

Solution We will first consider $y_1(x)$. It is clear that $y = x^3$ satisfies $xy' = 3y$ for $y \in (0, \infty)$ and $y \in (-\infty, 0)$. We must determine if it satisfies the ODE at $x = 0$.
We use the definition of the derivative to calculate the derivative of $y_1(x)$ at 0. $y_1'(0) = \lim\limits_{x \to 0} \dfrac{x^3}{x} = 0$. Thus the differential equation is satisfied at all points in $\mathbb{R}$.

For the solution $y_2(x) = 0$, it clearly satisfies $xy' = 3y$.

There is no contradiction of the existence and uniqueness theorem because if put in the form specified by the theorem, $y' = 3y/x$, we see that there is a discontinuity on the RHS at $x = 0$. Thus the hypotheses of the theorem are not satisfied, so it doesn't apply.

23) Solve the following differential equation $(y^2-3xy-2x^2)dx+(xy-x^2)dy=0$

Solution

The equation is clearly not separable, so we test for exactness:

$$\frac{\partial}{\partial y}(y^2-3xy+2x^2)=2y-3x$$
$$\frac{\partial}{\partial x}(xy-x^2)=y-2x$$

The equation is not exact, so we try an integrating factor $\mu=\mu(x)$.

$$\frac{\partial}{\partial y}\mu\cdot(y^2-3xy+2x^2)=\mu\cdot(2y-3x)$$
$$\frac{\partial}{\partial x}\mu\cdot(xy-x^2)=\mu\cdot(y-2x)+\mu'\cdot(xy-x^2)$$

$$\mu\cdot(y-2x)+\mu'\cdot(xy-x^2)=\mu\cdot(2y-3x).$$

Thus, $\mu(x)=x$.

So, we obtain the equation $(y^2x-3x^2y-2x^3)dx+(x^2y-x^3)dy=0$

Integrating, $\Phi(x,y)=\int(y^2x-3x^2y-2x^3)dx=\frac{x^2y^2}{2}-x^3y-\frac{x^4}{2}+f(y)$.
Differentiating with respect to y, we find $\Phi_y(x.y)=yx^2-x^3+f'(y)$. Thus $f'(y)=0$.

Multiplying through by 2 (to make it look nicer), we find as the solution,

$$x^2y^2-2x^3y-x^4=C.$$

24) Solve the following differential equation $x^2+y^2+x+xyy'=0$

Solution

The equation is clearly not separable, so we test for exactness:

$$\frac{\partial}{\partial y}(x^2+y^2+x)=2y$$

$$\frac{\partial}{\partial x}(xy)=y.$$

It is not exact, so we try an integrating factor $\mu=\mu(x)$.

We find an integrating factor of $\mu(x)=x$.

We obtain the new equation: $x^3+xy^2+x^2+x^2yy'=0$
Integrating, $\Phi(x,y)=\int x^3+xy^2+x^2dx=\frac{x^4}{4}+\frac{x^2y^2}{2}+\frac{x^3}{3}+f(y)$

Differentiating with respect to y, $\Phi_y(x,y)=x^2y+f'(y)$. Thus $f'(y)=0$.

Multiplying through by 12 to make it look nicer, we obtain the solution:

$$3x^4+6x^2y^2+4x^3=C$$

25) Solve the initial value problem: $e^x(y^3+xy^3+1)dx+3y^2(xe^x-6)dy=0$, $y(0)=1$

Solution

The equation is clearly not separable, so we test for exactness:

$$\frac{\partial}{\partial y}e^x(y^3+xy^3+1)=3y^2e^x(1+x)$$

$$\frac{\partial}{\partial x}3y^2(xe^x-6)=3y^2(1+x)e^x$$

The equation is exact.

Integrating, $\Phi(x,y)=\int e^x(y^3+xy^3+1)dx=e^x(1+xy^3)+f(y)$

Differentiating with respect to y, we obtain $\Phi_y(x,y)=3y^2xe^x+f'(y)=3y^2(xe^x-6)$.

Thus, $f'(y)=-18y^2\Rightarrow f(y)=-6y^3$.

Thus, $\Phi(x,y)=e^x(1+xy^3)-6y^3=C$

Plugging in the initial condition, $1-6=C=-5$

The final solution is then, $e^x+e^xxy^3-6y^3+5=0$