EUS Tutoring Session Review Problem Set **SOLUTIONS**

Mathematics 256 - Midterm 1

Differential Equations

Note on notation: Whenever $\log(x)$ is used without a subscript to indicate the base, it is assumed to be base e in math courses. Thus in this review package, $\log(x)$ and $\ln(x)$ are used interchangeably. For inverse trigonometric functions, $\sin^{-1}(x) = \arcsin(x)$, and the other inverse trigonometric functions are similarly denoted.

The solutions to these problems will be posted on ubcengineers.ca → Services → Academic Services → Tutoring. If you believe that there is an error in an answer key, or if you have suggestions for improvement of EUS tutoring sessions, please e-mail us at: tutoring@ubcengineers.ca.

The contents of this package include: Slope Fields, Separable Equations, First Order Linear Equations, Existence and Uniqueness Theorem, Wronskians, Second order linear constant coefficient Equations, Reduction of Order.

1) Solve the following differential equation for $y = y(x)$. $\dfrac{dy}{dx} - \dfrac{2xy}{x^2+1} = 1$

Solution

Find the integrating factor $\mu(x) = e^{\int \frac{-2x}{x^2+1}dx} = e^{-\log(x^2+1)} = \dfrac{1}{x^2+1}$.

Multiply both sides of the equation by the integrating factor → $\dfrac{1}{x^2+1}\dfrac{dy}{dx} - \dfrac{2x}{(x^2+1)^2}y = \dfrac{1}{x^2+1} = \left(\dfrac{y}{1+x^2}\right)'$

Integrating, $\dfrac{y}{x^2+1} = \arctan x + C \Rightarrow y = (x^2+1)\arctan x + C(x^2+1)$

2) Solve the following differential equation for $r = r(\theta)$. $\tan\theta \dfrac{dr}{d\theta} - r = \tan^2\theta$

Solution

Put this in the form in which we will use the integrating factor $\mu(\theta)$.
$r' - r \cdot \cot\theta = \tan\theta$.

$\mu = e^{\int -\cot\theta d\theta} = e^{-\log\sin x} = \csc\theta$

Differential equation will become $r'\csc\theta - r\csc\theta\cot\theta = \sec\theta = (r\csc\theta)'$

Integrating, $\log(\sec\theta + \tan\theta) + C = r\csc\theta$

Thus, $r(\theta) = C\sin\theta + \sin\theta\log(\sec\theta + \tan\theta)$

3) Solve the following differential equation explicitly as a function of x. $(y^2+1)dx - (x^2+1)dy = 0$

Solution

Separate variables: $\dfrac{dy}{y^2+1} = \dfrac{dx}{x^2+1}$

Integrate: $\arctan y = \arctan x + C \Rightarrow y = \tan(\arctan x + C) = \dfrac{x + C'}{1 - C'x}$.

4) Show that the function

$$y_1(x) = \begin{cases} 0, & x < 0 \\ x^3, & x \geq 0 \end{cases}$$

is a solution of the initial value problem $xy' = 3y$, $y(0) = 0$. Show that $y_2(x) \equiv 0$ is a second solution. Explain why this does not contradict the existence and uniqueness theorem.

Solution

We will first consider $y_1(x)$. It is clear that $y = x^3$ satisfies $xy' = 3y$ for $y \in (0, \infty)$ and $y \in (-\infty, 0)$. We must determine if it satisfies the ODE at $x = 0$.
We use the definition of the derivative to calculate the derivative of $y_1(x)$ at 0.

$y_1'(0) = \lim_{x \to 0} \dfrac{x^3}{x} = 0$. Thus the differential equation is satisfied at all points in $\mathbb{R}$.

For the solution $y_2(x) = 0$, it clearly satisfies $xy' = 3y$.

There is no contradiction of the existence and uniqueness theorem because if put in the form specified by the theorem, $y' = 3y/x$, we see that there is a discontinuity on the RHS at $x = 0$. Thus the hypotheses of the theorem are not satisfied, so it doesn't apply.

5) Determine whether or not the initial value problem $y' = \cos(x + y)$, $y(x_0) = y_0$ has a unique solution defined on all of $\mathbb{R}$.

Solution

Let $f(x, y) = \cos(x + y)$. $f(x, y)$ is continuous everywhere, and $f_y(x, y) = -\sin(x + y)$ is also continuous everywhere. Thus, by the existence and uniqueness theorem, the initial value problem given has a unique solution defined on all of $\mathbb{R}$.

6) Solve the following differential equation. Express the solution in terms of k. $y'' + 4ky - 12k^2y = 0$

Solution

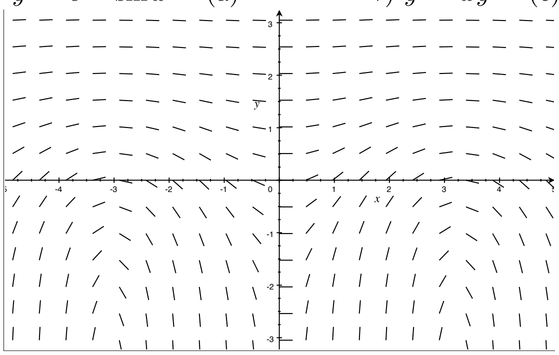
The characteristic equation is $r^2 + 4rk - 12k^2 = 0$. The roots are $r_1 = -6k$ and $r_2 = 2k$.

Thus the general solution is $y(x) = c_1e^{-6kx} + c_2e^{2kx}$.

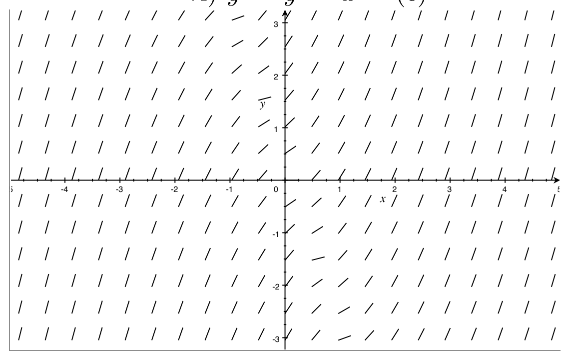
7) Match the following differential equations with their slope fields.

i) $y' = 4x - 2y/x =$ (d) ii) $y' = \sqrt{|3x + y|} =$ (b) iii)$y' = 1 + 2xy =$ (f)

iv) $y' = e^{-y} \sin x =$ (a) v) $y' = xy =$ (c) vi) $y' = y^2 - x =$ (e)



a)



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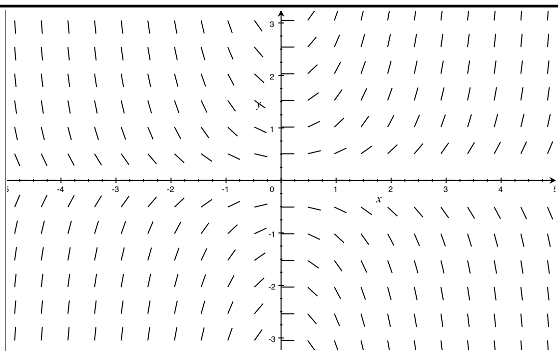

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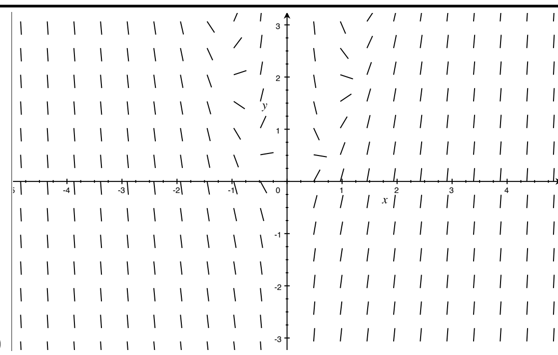

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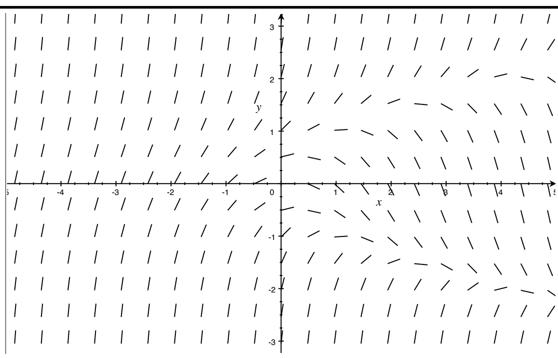

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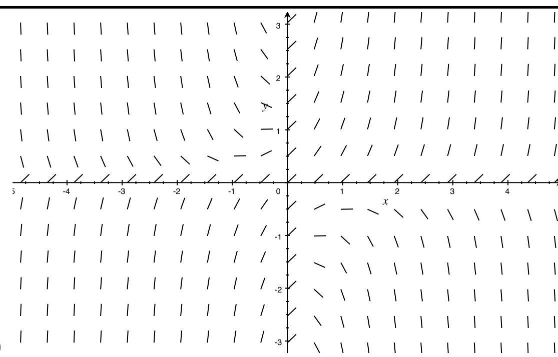


f)

8) Solve the equation. $\dfrac{dr}{d\theta} = (r + e^{-\theta})\tan\theta$

Solution

We will rearrange this into the first order linear form: $r' - r\tan\theta = e^{-\theta}\tan\theta$

The integrating factor will be $\mu(x) = e^{\int -\tan\theta d\theta} = e^{\log\cos\theta} = \cos\theta$

Multiplying through by the integrating factor, $r'\cos\theta - r\sin\theta = e^{-\theta}\sin\theta = (r\cos\theta)'$

Integrating, $r\cos\theta = \displaystyle\int e^{-\theta}\sin\theta d\theta = \frac{-e^{-\theta}}{2}(\cos\theta + \sin\theta) + C$

$r(\theta) = \dfrac{-e^{-\theta}}{2}(1 + \tan\theta) + C\sec\theta$

9) Solve the equation $\tan\theta\dfrac{dr}{d\theta} - r = \tan^2\theta$

Solution

Put this in the form in which we will use the integrating factor $\mu(\theta)$.
$r' - r\cdot\cot\theta = \tan\theta.$

$\mu = e^{\int -\cot\theta d\theta} = e^{-\log\sin x} = \csc\theta$

Differential equation will become $r'\csc\theta - r\csc\theta\cot\theta = \sec\theta = (r\csc\theta)'$

Integrating, $\log(\sec\theta + \tan\theta) + C = r\csc\theta$

Thus, $r(\theta) = C\sin\theta + \sin\theta\log(\sec\theta + \tan\theta)$

10) Solve the equation. $xy' - y = x^2\sin x$

Solution

Rearranging into standard form, $y' - \dfrac{y}{x} = x\sin x$

The integrating factor is $\mu(x) = e^{\int -1/x dx} = e^{-\log x} = \frac{1}{x}$
Multiplying through, $\dfrac{y'}{x} - \dfrac{y}{x^2} = \sin x = \left(\dfrac{y}{x}\right)'$

Integrating, $\left(\dfrac{y}{x}\right) = -\cos x + C$

$y(x) = -x\cos x + Cx.$

11) Solve the equation. $x\frac{dy}{dx} + 2y = \frac{-\sin x}{x}$

Solution

Rearange the equation into standard form: $\frac{dy}{dx} + \frac{2}{x}y = \frac{-\sin x}{x^2}$

Find the integrating factor: $\mu(x) = e^{\int \frac{2}{x}dx} = e^{\log x^2} = x^2$

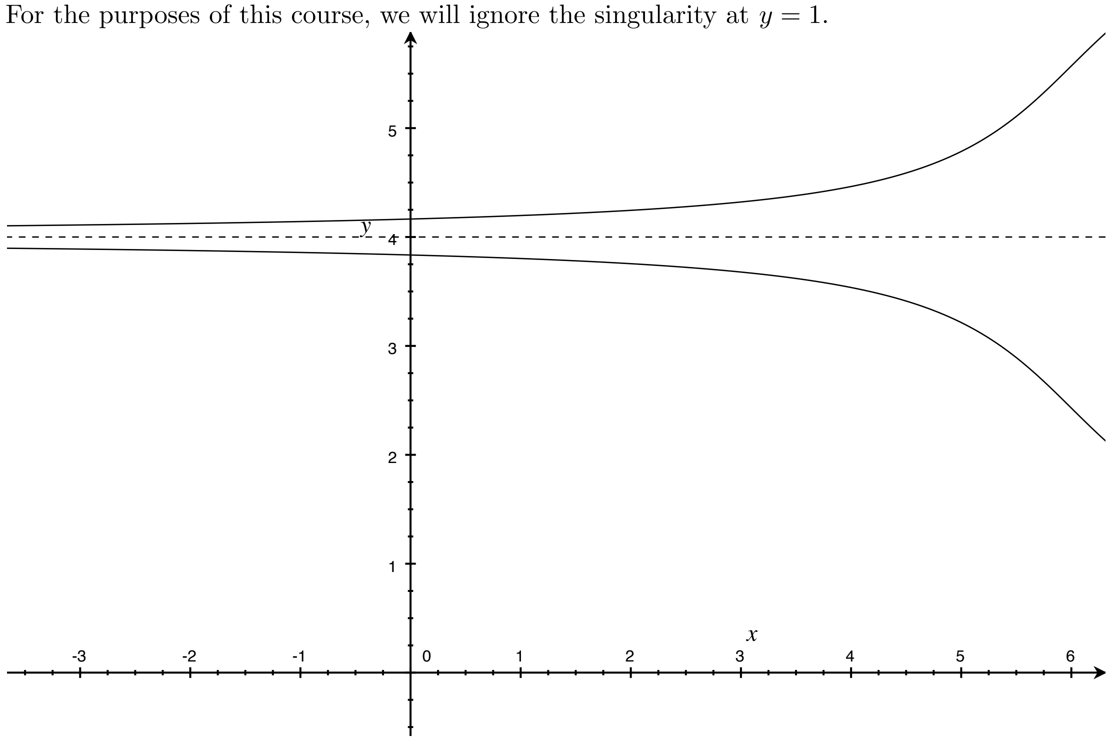
Multiplying through, $x^2\frac{dy}{dx} + 2xy = -\sin x = (x^2y)'$

Integrating, $(yx^2) = \cos x + C \Rightarrow y = x^{-2}\cos x + Cx^{-2}$

12) Create a phase diagram for the following autonomous equation. $y' = \frac{y-4}{y-1}$

Solution

For the purposes of this course, we will ignore the singularity at $y = 1$.

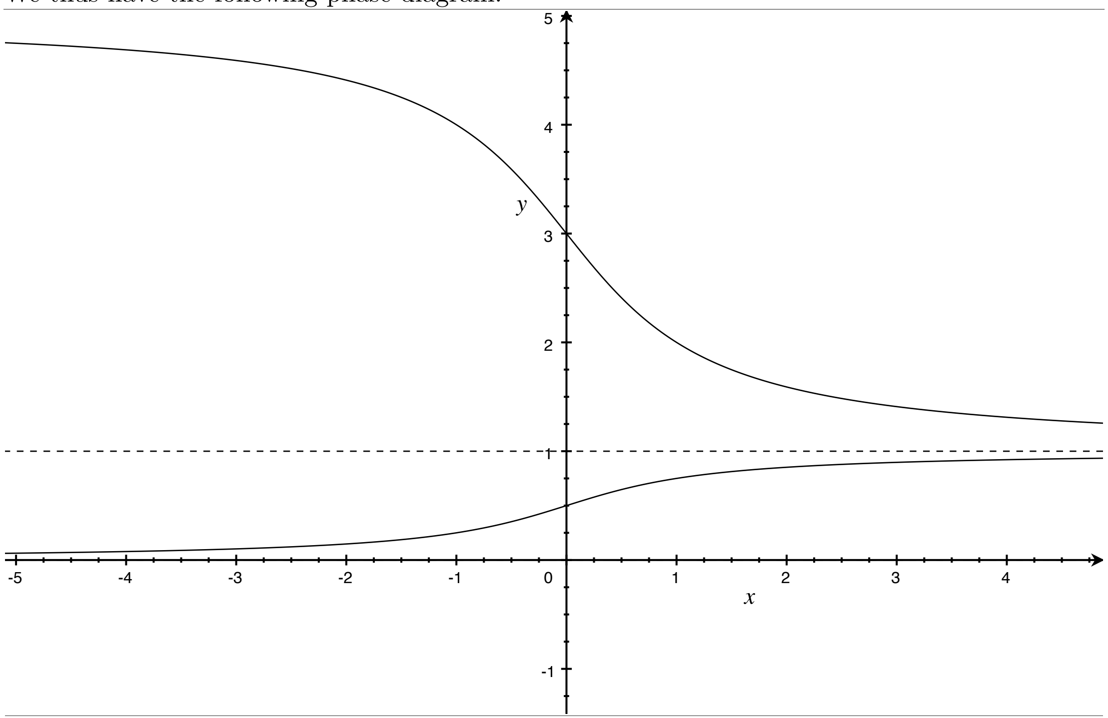


We have a source at $y = 4$.

13) Create a phase diagram for the following autonomous equation. $y' = \sqrt{y} - y^2$

Solution

The critical points are at $y = 0, 1$.
For $y \in (0, 1)$, $y' > 0$
For $y \in (1, \infty)$, $y' < 0$
We thus have the following phase diagram:

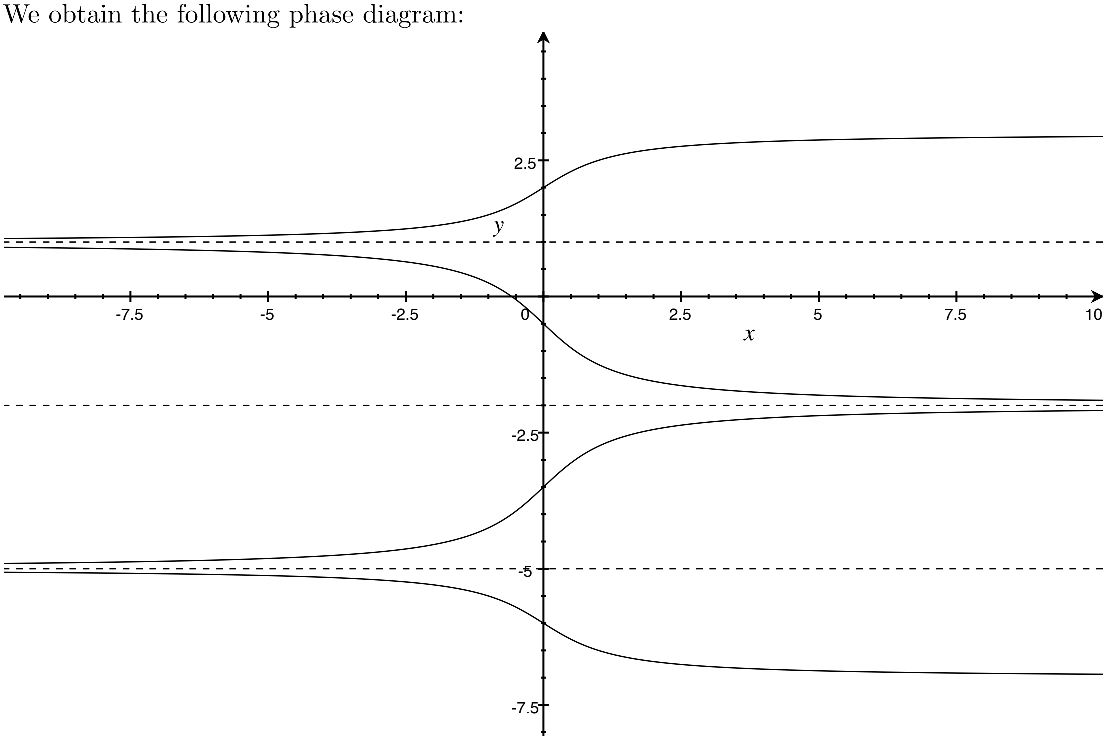


14) Create a phase diagram for the following autonomous equation. $y' = y^3 + 6y^2 + 3y - 10$

Solution

We factor $y' = y^3 + 6y^2 + 3y - 10 = (y + 5)(y + 2)(y - 1)$

For $y \in (-\infty, -5)$, $y' < 0$
For $y \in (-5, -2)$, $y' > 0$
For $y \in (-2, 1)$, $y' < 0$
For $y \in (1, \infty)$, $y' > 0$
We obtain the following phase diagram:



15) Create a phase diagram for the following autonomous equation for $y \in [-2\pi, 2\pi]$. $y' = \sin y \log y$

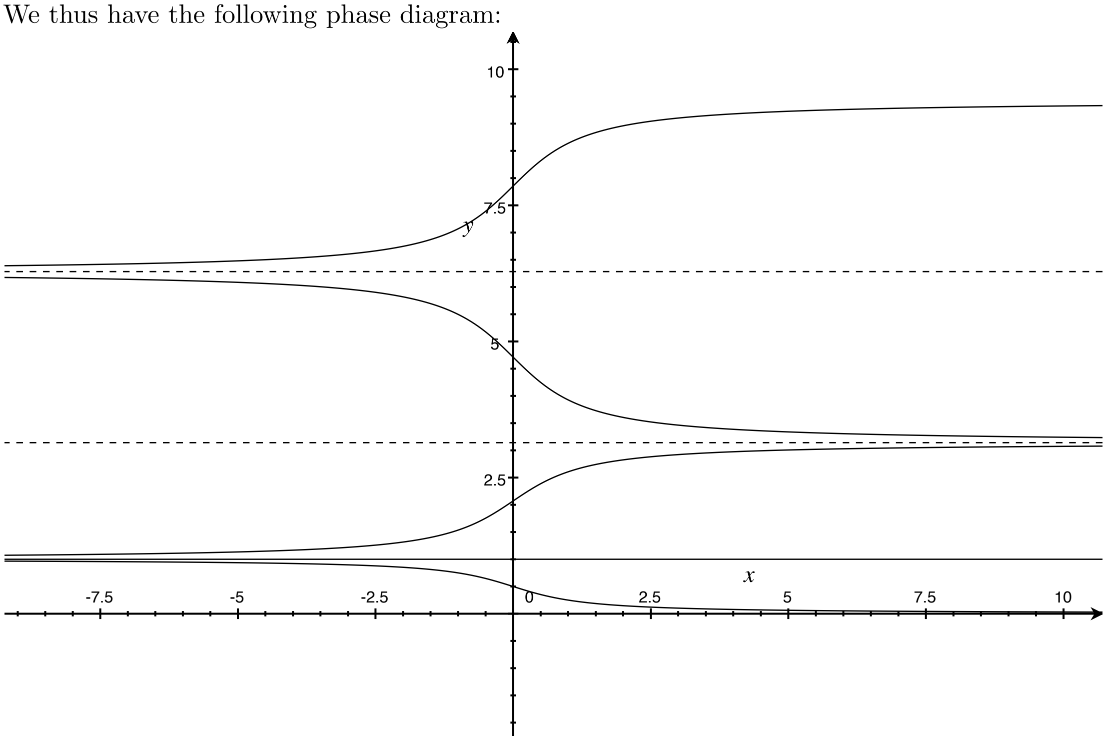
Solution

Even though the problem says $y \in [-2\pi, 2\pi]$, we must restrict it further because the logarithm only accepts positive values.

First of all, we must see if y' will approach 0 as y approaches 0.

It is left as an exercise (L'Hopital's Rule) to compute $\lim_{y \to 0} \sin y \log y = 0$
We now compute the values of $f(y) = \sin y \log y$ at points between critical points.
The critical points are: $y \in \{1, \pi, 2\pi\}$.
For $y \in (0, 1)$, $y' < 0$
For $y \in (1, \pi)$, $y' > 0$
For $y \in (\pi, 2\pi)$, $y' < 0$
For $y \in (2\pi, 3\pi)$, $y' > 0$

We thus have the following phase diagram:



16) Solve the following differential equation. $x \log x dy + \sqrt{1+y^2} dx = 0$

Solution

The equation is separable, so we rearrange to obtain: $\dfrac{-dy}{\sqrt{1+y^2}} = \dfrac{dx}{x \log x}$.

Integrating, $\log(\sqrt{1+y^2} - y) = \log \log x + C$.

17) Solve the following differential equation. $e^{x+1}\tan y dx + \cos y dy = 0$

Solution

Separate variables: $e^{x+1}dx = -\cot y \cos y dy = -\dfrac{\cos^2 y}{\sin y}dy = -\dfrac{1-\sin^2 y}{\sin y}dy = (\sin y - \csc y)dy$

Integrating, $-\cos y + \log|\csc y + \tan y| = e^{x+1} + C$

18) Assume that $y_p(x) = x^2$ is a solution of $y'' + y' - 2y = 2(1 + x - x^2)$. Find a particular solution to $y'' + y' - 2y = 6(1 + x - x^2)$.

Solution

By linearity, the new particular solution $y_q(x) = 3x^2$.

19) Assume that $y_1 = 1 + x$ is a solution of $y'' - y' + y = x$, and $y_2 = e^{2x}$ is a solution of $y'' - y' + y = 3e^{2x}$. Find a particular solution to $y'' - y' + y = x + 3e^{2x}$.

Solution

By linearity, $y_p = 1 + x + e^{2x}$.

20) Find the general solution to $y''' + y'' - 6y' = 0$.

Solution

We can make the substitution $y' = z$ to reduce this to a second order equation.

$z'' + z' - 6z = 0$. Then the characteristic equation is $r^2 + r - 6 = 0$. The roots are $r = -3$, and $r = 2$.

Thus, $z = c_1e^{-3x} + c_2e^{2x} = y'$

So, $y(x) = c_1'e^{-3x} + c_2'e^{2x} + c_3$

21) Find the general solution to $u'' - 2au' + a^2u = 0$

Solution

The characteristic equation is $r^2 - 2ar + a^2 = 0$. The roots are $r_1 = a$ and $r_2 = a$. We have repeated roots, so therefore $y = c_1e^{ax} + c_2xe^{ax}$.

22) For the given differential equation, use Euler's Method with step size $\frac{1}{2}$ to estimate $y(2)$ if the solution passes through (1,0). $\dfrac{dy}{dx} = x - \dfrac{y^2}{4}$

Solution

Let $f(x, y) = x - \dfrac{y^2}{4}$

$y(3/2) \approx y(1) + f(1, y(1))(1/2) = 1/2$
$y(2) \approx y(3/2) + f(3/2, y(3/2))(1/2) = 7/16$

23) Use Euler's Method with step size 0.2 to estimate $y(1)$, where $y(x)$ is the solution of the initial value problem $y' = xy - x^2$, $y(0) = 1$.

Solution

Let $f(x, y) = xy - x^2$
$y(0.2) \approx y(0) + f(0, y(0))(0.2) = 1$
$y(0.4) \approx y(0.2) + f(0.2, y(0.2))(0.2) = 29/25$
$y(0.6) \approx y(0.4) + f(0.4, y(0.4))(0.2) = 763/625$
$y(0.8) \approx y(0.6) + f(0.6, y(0.6))(0.2) = 1.369792$
$y(1) \approx y(0.8) + f(0.8, y(0.8))(0.2) = 1.461$

24) If $y_1(x) = x$ is a solution to the differential equation $y'' + (x^2 - x)y' + (1 - x)y = 0$, find a second solution $y_2(x)$

Solution

Form a new solution as $y_2(x) = x \cdot v(x)$, and plug into the ODE.
We obtain, $2v' + xv'' + (x^2 - x)(xv' + v) + (1 - x)v = 2v' + xv'' + x^3v' + x^2v - x^2v' - xv + xv - x^2v$

$(2 - x^2 + x^3) + x\dfrac{v''}{v'} = 0 \Rightarrow \dfrac{2}{x} - x + x^2 + (\log(v'))' = 0.$

Integrating, $2\log x - x^2/2 + x^3/3 + \log(v') = C$

Solving for $v' = c_1\dfrac{e^{x^2/2 - x^3/3}}{x^2}$

Thus, $v(x) = c_2 + c_1\displaystyle\int \dfrac{e^{x^2/2 - x^3/3}}{x^2}dx$

$$y_2(x) = xv = c_2x + c_1x\int \frac{e^{x^2/2 - x^3/3}}{x^2}dx$$

25) If $y_1(x) = e^x$ is a solution to the differential equation $y'' + (x^2 - 1)y' - x^2y = 0$, find a second solution $y_2(x)$

Solution

Let $y_2(x) = e^xv(x)$. We will plug this into the original equation.

$e^x(v'' + 2v' + v) + (x^2 - 1)e^x(v' + v) - x^2e^xv = 0$

Dividing through by e^x, we obtain $v'' + v'(1 + x^2) = 0$.

Thus, $\log v' = -\arctan x + C_1 \Rightarrow v' = C_1'e^{-\arctan x}$

$v(x) = C_1'\displaystyle\int e^{-\arctan x}dx + C_2$

$$y_2(x) = e^xv(x) = e^xC_2 + C_1'e^x\int e^{-\arctan x}dx$$

26) Show that each of the functions $y_1 = \sin x - \frac{1}{3}\sin 3x$ and $y_2 = \sin^3 x$ is a solution to $y'' + (\tan x - 2\cot x)y' = 0$, but that $y = c_1y_1 + c_2y_2$ is *not* the general solution of the differential equation.

Solution

We will show that y_1 is a solution to the ODE.

$$\begin{aligned}
y'' + (\tan x - 2\cot x)y' &= -\sin x + 3\sin 3x + (\tan x - 2\cot x)(\cos x - \cos 3x) \\
&= -\sin x + 3\sin 3x + \tan x\cos x - \tan x\cos 3x + 2\cot x\cos 3x - 2\cot x\cos x \\
&= 3\sin 3x - \tan x\cos 3x + 2\cot x\cos 3x - 2\cot x\cos x \\
&= 3\sin(x+2x) - \tan x\cos(x+2x) + 2\cot x\cos(x+2x) - 2\cot x\cos x \\
&= 3[\sin x\cos 2x + \cos x\sin 2x] - \tan x\cos(x+2x) + 2\cot x\cos(x+2x) - 2\frac{\cos^2 x}{\sin x} \\
&= 3[\sin x\cos 2x + \cos x\sin 2x] + (\cos x\cos 2x - \sin x\sin 2x)(2\cot x - \tan x) - 2\frac{\cos^2 x}{\sin x} \\
&= 3[\sin x\cos 2x + \cos x\sin 2x] + (\cos x\cos 2x - \sin x\sin 2x)\frac{\cos^2 x + \cos 2x}{\sin x\cos x} - 2\frac{\cos^2 x}{\sin x}
\end{aligned}$$

It is left as an exercise to use the double angle identities to show that this reduces to 0.

We will show that y_2 is a solution to the ODE.
$6\sin x\cos^2 x - 3\sin^3 x + (\tan x - 2\cot x)(3\sin^2 x\cos x) = 0$

In order to prove that this does not form a general solution, we compute the Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin x - \frac{1}{3}\sin 3x & \cos x - \cos 3x \\ \sin^3 x & 3\sin^2 x\cos x \end{vmatrix}$$

$$\begin{aligned}
W &= \left(\sin x - \frac{1}{3}\sin 3x\right)(3\sin^2 x\cos x) - (\cos x - \cos 3x)(\sin^3 x) \\
&= 2\sin^3 x\cos x - \sin 3x\sin^2 x\cos x + \sin^3 x\cos 3x \\
&= 2\sin^3 x\cos x - (\sin x\sin 2x + \cos x\cos 2x)\sin^2 x\cos x + \sin^3 x(\cos x\cos 2x - \sin x\sin 2x) \\
&= 0
\end{aligned}$$

The Wronskian is 0, so the solutions are linearly *dependent* and thus cannot form a general solution.