EUS Review Problem Set **Solutions**

Mathematics 256 - Midterm 2

Differential Equations

Note on notation: Whenever $\log(x)$ is used without a subscript to indicate the base, it is assumed to be base e in math courses. Thus in this review package, $\log(x)$ and $\ln(x)$ are used interchangeably. For inverse trigonometric functions, $\sin^{-1}(x) = \arcsin(x)$, and the other inverse trigonometric functions are similarly denoted.

The solutions to these problems will be posted on ubcengineers.ca → Services → Academic Services → Tutoring. If you believe that there is an error in an answer key, or if you have suggestions for improvement of EUS tutoring sessions, please e-mail us at: tutoring@ubcengineers.ca.

The contents of this package include: Second order linear equations with undetermined coefficients, variation of parameters, Laplace transforms.

Special thanks to Rahat Dhande and Aaron Troy for preparing the solution set.

1) Solve the initial value problem. $y'' - 5y' + 6y = e^x(2x-3)$, $y(0) = 1$, $y'(0) = 3$.

Solution

The characteristic equation for this differential equation is $r^2 - 5t + 6 = 0$. This yields roots of $r = 2, 3$. Thus the complementary solutions will be $y_c(x) = c_1e^{2x} + c_2e^{3x}$.

Now we can guess a solution and use undetermined coefficients in order to find the particular solution. Note that we should not plug in the initial conditions to solve for c_1 and c_2 until we have the particular solution as well.

Let $y_p(x) = e^x(Ax+B)$, $y_p'(x) = e^x(Ax+B+A)$, and $y_p''(x) = e^x(Ax+B+2A)$

Plugging these back into the differential equation we obtain:
$e^x(Ax+b+2a) - 5e^x(Ax+B+A) + 6e^x(Ax+B) = e^x(2x-3)$

We now divide by e^x and sum coefficients to find that $A = 1$ and $B = 0$
Therefore $y_p(x) = xe^x$

So we have: $y(x) = y_c(x) + y_p(x) = c_1e^2x + c_2e^3x + xe^x$

Since $y(0) = 1 = c_1 + c_2$ and $y'(0) = 3 = 2c_1 + 3c_2 + 1$ we can see that $c_1 = 1$ and $c_2 = 0$

So the general solution is $y(x) = e^{2x} + xe^x = e^x(e^x + x)$

2) Solve the equation $y'' + y = \sec x$

Solution

The characteristic equation is $r^2 + 1 = 0$. This yields the roots $r = \pm i$

Therefore the complimentary solution is: $y_c(x) = c_1\cos(x) + c_2\sin(x)$

Now we can guess a solution and use variation of parameters.

Let $y_p(x) = u_1(x)\cos(x) + u_2(x)\sin(x)$, such that $u_1'(x)\cos(x) + u_2'(x)\sin(x) = 0$

$y'' + y = \sec(x)$ therefore simplifies to $-u_1'(x)\sin(x) + u_2'(x)\cos(x) = \sec(x)$

Using the condition $y_c(x) = u_1\cos(x) + u_2\sin(x)$, we find $u_1' = -\tan(x)$ and $u_2' = 1$

Through integration $u_1 = \ln|\cos(x)|, u_2 = x$

We have no initial conditions, so the general solution is: $c_1\cos(x) + c_2\sin(x) + \ln|\cos(x)|\cos(x) + x\sin(x)$

3) Solve the equation $y'' + y' + y = x^2$

Solution

The characteristic equation is $r^2 + r + 1 = 0$, this yields the roots $r = \frac{-1}{2} \pm \frac{\sqrt{3}i}{2}$

Therefore the complimentary solution is: $y_c(x) = e^{-x/2}\left(c_1 cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 sin\left(\frac{\sqrt{3}}{2}x\right)\right)$
We can now guess a particular solution.

Let $y_p(x) = Ax^2 + Bx + C$. Plugging this back into our original differential equation and simplifying, we find $A = 1, B = -2, C = 0$

So $y_p(x) = x^2 - 2x$

Therefore the general solution is $y(x) = e^{-x/2}\left(c_1 cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 sin\left(\frac{\sqrt{3}}{2}x\right)\right) + x^2 - 2x$

4) Solve the equation $y'' - 3y' + 2y = \cos(e^x)$

Solution

The characteristic equation is $r^2 - 3r + 2 = 0$, this yields the roots $r = 1, 2$.

Therefore the complimentary solution is $y_c(x) = c_1e^x + c_2e^{2x}$
We can now guess a particular solution and use variation of parameters.

Let $y_p(x) = u_1(x)e^x + u_2(x)e^{2x}$ such that $u_1'(x)e^x + u_2'(x)e^{2x} = 0$

Plugging this into $y_p'' - 3y_p' + 2y = cos(e^x)$ we find and using the condition $u_1'(x)e^x + u_2'(x)e^{2x} = 0$, we find that $u_1'(x) = -e^{-x}\cos(e^x)$ and $u_2'(x) = e^{-2x}\cos(e^x)$

We then integrate to get $u_1(x) = -\int e^{-x}\cos(e^x)\,dx$ and $u_2(x) = \int e^{-2x}\cos(e^x)\,dx$

Plugging these back into our particular solution, we find $y_p(x) = -e^x\int e^{-x}\cos(e^x)\,dx + e^{2x}\int e^{-2x}\cos(e^x)\,dx$

Therefore the general solution is $y(x) = c_1e^x + c_2e^{2x} - e^x\int e^{-x}\cos(e^x)\,dx + e^{2x}\int e^{-2x}\cos(e^x)\,dx$

5) Solve the equation $y'' + 2y' + y = \dfrac{e^{-x}}{x}$

Solution

The characteristic equation is $r^2 + 2r + 1$, this yields the root $r = -1$

Because there is a single real root, the complimentary solution is $y_c(x) = c_1 e^{-x} + c_2 x e^{-x}$

Let $y_p(x) = u_1(x) e^{-x} + u_2(x) x e^{-x}$ such that $u_1'(x) e^{-x} + u_2'(x) x e^{-x} = 0$

Plugging this into $y_p'' + 2y_p' + y = \frac{e^{-x}}{x}$ and using the condition $u_1'(x) x^e + u_2'(x) e^{2x} = 0$, we find that $u_1'(x) = -1$ and $u_2'(x) = \frac{1}{x}$

Integrating, we get $u_1(x) = -x$ and $u_2(x) = ln|x|$

So the particular solution is $y_p(x) = -xe^{-x} + ln|x|xe^{-x}$

Therefore the general solution is $y(x) = y_c(x) + y_p(x) = c_1 e^{-x} + c_2 x e^{-x} + x ln|x| e^{-x}$

Note that we have combined the second term of $y_c(x)$ and the first term of $y_p(x)$ since $c_2 - 1$ is also a constant.

6) Solve the equation with the two solutions of the homogeneous equation given. $x^2 y'' - xy' + y = x$, $y_1(x) = x$, $y_2(x) = x \log x$.

Solution

Begin by dividing the equation by x^2 to obtain $y'' + \dfrac{1}{x} y' + \dfrac{1}{x^2} y = \dfrac{1}{x}$

Using the given solutions, the complimentary solution is: $y_c(x) = c_1 x + c_2 x \log x$

Let $y_p(x) = u_1(x) x + u_2(x) x \log x$ such that $u_1'(x) x + u_2'(x) \log x = 0$

Plugging this into the equation $y'' + \dfrac{1}{x} y' + \dfrac{1}{x^2} y = \dfrac{1}{x}$ and using the condition $u_1'(x) x + u_2'(x) \log x = 0$ we find that $u_1'(x) = -\dfrac{\log x}{x}$ and $u_2'(x) = \dfrac{1}{x}$

We now integrate so that $u_1(x) = -\displaystyle\int \log(x) \frac{1}{x} dx = \frac{-(\log x)^2}{2}$, $u_2(x) = \displaystyle\int \frac{1}{x} dx = \log|x|$

Therefore, $y_p(x) = \frac{-(\log x)^2}{2} x + \log|x| x \log x = \frac{(\log x)^2 x}{x}$, $x > 0$

So the general solution is $y(x) = y_c(x) + y_p(x) = c_1 x + c_2 x \log x + \dfrac{-(\log x)^2}{2} x + x \log|x| \log x$
$= \dfrac{(\log x)^2 x}{2} + c_1 x + c_2 x \log x,\ x > 0$

7) Solve the equation $y'' + 3y' + 2y = 8 + 6e^x + 2\sin x$

Solution

The characteristic equation is $r^2 + 3r + 2 = 0$. This yields the roots $r = -1, -2$

So the complimentary solution is $y_c(x) = c_1 e^{-x} + c_2 e^{-2x}$

For this problem, we will have to find three different particular solutions, one for each term on the RHS.

Let $y''_{p1}(x) + 3y'_{p1}(x) + 2y_{p1}(x) = 8$

The RHS is just a constant, so $y_{p1} = A$, $2A = 8$, $y_{p1} = 4$

Let $y''_{p2}(x) + 3y'_{p2}(x) + 2y_{p2}(x) = 6e^x$, $y_{p2}(x) = Be^x$

We now find $Be^x(1+3+2) = 6e^x$

$B = 1$, $y_{p2}(x) = e^x$

Let $y''_{p3}(x) + 3y'_{p3}(x) + 2y_{p3}(x) = 2\sin(x)$, $y_{p3}(x) = C\cos(x) + D\sin(x)$

Solving for our constants we find: $C = -\dfrac{3}{5}$, $D = \dfrac{1}{5}$

So $y_{p3}(x) = -\dfrac{3\cos(x)}{5} + \dfrac{\sin(x)}{5}$

So the general solution is: $y(x) = y_c(x) + y_{p1}(x) + y_{p2}(x) + y_{p3}(x)$

$y(x) = c_1e^{-x} + c_2e^{-2x} + 4 + e^x - \dfrac{3\cos(x)}{5} + \dfrac{\sin(x)}{5}$

8) Solve the equation $y'' + 2y' + y = x^2e^{-x}$

Solution

The characteristic equation is $r^2 + 2r + 1 = 0$. This yields the root $r = -1$

So the complimentary solution is $y_c(x) = c_1e^x + c_2xe^x$. Note the special form due to a single real root.

Let $y_p(x) = x^2e^{-x}\left(Ax^2 + Bx + C\right) = e^{-x}\left(Ax^4 + Bx^3 + Cx^2\right)$

By differentiating and substituting into the original differential equation we find $e^{-x}\left(Ax^2 + Bx + C\right) = e^{-x}x^2$

Solving for the coefficients, $A = \frac{1}{12}, B = 0, C = 0$

So $y_p(x) = \frac{x^4}{12}e^{-x}$

The general solution is then $y(x) = c_1e^x + c_2xe^x + \frac{x^4}{12}e^{-x}$

9) Solve $y'' + y = \sin(2x)\sin(x)$. The following identity may be helpful: $\sin(2x)\sin(x) = \frac{1}{2}\cos(x) - \frac{1}{2}\cos(3x)$.

Solution

Begin by using the identity $\sin(2x)\sin(x) = \dfrac{1}{2}\cos(x) - \dfrac{1}{2}\cos(3x)$ to rewrite the equation as $y'' + y = \dfrac{1}{2}\cos(x) - \dfrac{1}{2}\cos(3x)$

The characteristic equation is $r^2 + 1 = 0$, this yields the roots $r = \pm i$

Therefore, the complimentary solution is $y_c(x) = c_1 \cos(x) + c_2 \sin(x)$

For this differential equation, we need to find two particular solutions.

Let $y''_{p1}(x) + y_{p1}(x) = \frac{1}{2}\cos(x)$, $y_{p1}(x) = x\left(A\cos(x) + B\sin(x)\right)$

By deriving and substituting into the differential equation, we find $-2A\sin(x) - 2B\cos(x) = \frac{1}{2}\cos(x)$

Solving for the coefficients: $A = 0, B = -\dfrac{1}{4}$, So $y_{p1}(x) = -\dfrac{x}{4}\sin(x)$

Let $y''_{p2}(x) + y_{p2}(x) = -\frac{1}{2}\cos(3x)$, $y_{p2} = A\cos(3x) + B\sin(3x)$

Deriving and substituting into the differential equation we find the coefficients: $A = \dfrac{1}{16}, B = 0$

So $y_{p2}(x) = \dfrac{\cos(3x)}{16}$

The general solution is then $y(x) = c_1\cos(x) + c_2\sin(x) - \dfrac{x}{4}\sin(x) + \dfrac{\cos(3x)}{16}$

10) Find the inverse Laplace transform of $F(s) = \dfrac{e^{-s}(s-1)}{s}$

Solution

$\mathcal{L}^{-1}[F(s)](t) = \mathcal{L}^{-1}[e^{-s}](t) - \mathcal{L}^{-1}\left[\frac{e^{-s}}{s}\right](t)$
$\mathcal{L}^{-1}[F(s)](t) = s(t-1) - u(t-1)$

11) Find the inverse Laplace transform of $F(s) = \dfrac{s+10}{s^3 + 2s^2 + 10s}$

Solution

We first use partial fraction decomposition to rewrite $F(s)$:

$$\frac{s+10}{s^3 + 2s^2 + 10s} = \frac{s+10}{s(s^2 + 2s + 10)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 1}$$

$A = 1, B = -1, C = -1$

So: $F(s) = \dfrac{1}{s} - \dfrac{s+1}{s^2 + 2s + 10} = \dfrac{1}{s} - \dfrac{s}{(s+1)^2 + 9} - \dfrac{1}{(s+1)^2 + 9}$

$$\mathcal{L}^{-1}[F(s)](t) = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2 + 9}\right]$$

$$\mathcal{L}^{-1}[F(s)](t) = 1 - e^{-t}\mathcal{L}^{-1}\left[\frac{s}{s^2 + 9}\right]$$

$$\mathcal{L}^{-1}[F(s)](t) = 1 - e^{-t}\cos(3t)$$

12) Find the inverse Laplace transform of $F(s) = \dfrac{2s-1}{(4s^2 + 1)(9s^2 + 1)}$

Solution

Begin by rewriting $F(s)$ using partial fraction decomposition.

$$\frac{2s-1}{(4s^2+1)(9s^2+1)} = \frac{As+B}{4s^2+1} + \frac{Cs+D}{9s^2+1}$$

$$A = -\frac{8}{5}, B = \frac{4}{5}, C = \frac{18}{5}, D = -\frac{9}{5}$$

So $F(s) = \frac{4}{5}\left(\frac{-2s+1}{4s^2+1}\right) + \frac{9}{5}\left(\frac{2s-1}{9s^2+1}\right)$

Now $\mathcal{L}^{-1}[F(s)](t) = \frac{4}{5}\left(-\frac{1}{2}\mathcal{L}^{-1}\left[\frac{s}{s^2+1/4}\right] + \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2+1/4}\right]\right) + \frac{9}{5}\left(\frac{2}{9}\mathcal{L}^{-1}\left[\frac{s}{s^2+1/9}\right] - \frac{1}{9}\mathcal{L}^{-1}\left[\frac{1}{s^2+1/9}\right]\right)$

$$\mathcal{L}^{-1}[F(s)](t) = -\frac{2}{5}\cos\left(\frac{t}{2}\right) + \frac{2}{5}\sin\left(\frac{t}{2}\right) + \frac{2}{5}\cos\left(\frac{t}{3}\right) - \frac{3}{5}\sin\left(\frac{t}{3}\right)$$

13) Express the given function in terms of unit step functions, and then find its Laplace transform.

$$f(t) = \begin{cases} t^2 & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$$

Solution

Using the Heaviside function, we write $f(t) = t^2(u(t-0) - u(t-1)) + 0(u(t-1) - u(t-\infty))$

$f(t) = t^2(1 - u(t-1))$

$\mathcal{L}[f(t)](s) = \mathcal{L}[t^2] - \mathcal{L}[u(t-1)t^2]$

$\mathcal{L}[f(t)](s) = \frac{2}{s^3} - e^{s}\mathcal{L}\left[(t+1)^2\right]$

$\mathcal{L}[f(t)](s) = \frac{2}{s^3} - e^{-s}\left(\mathcal{L}[t^2] + 2\mathcal{L}[t] + \mathcal{L}[1]\right)$

$\mathcal{L}[f(t)](s) = \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} = \frac{2}{s^2} + \frac{1}{s}\right)$

14) Express the given function in terms of unit step functions, and then find its Laplace transform.

$$f(t) = \begin{cases} 2t-1 & 0 \le t < 2 \\ t & t \ge 2 \end{cases}$$

Solution

Using the Heaviside function, we write $f(t) = (2t-1)(u(t-0) - u(t-2)) + t(u(t-2) - u(t-\infty))$

Simplifying, $f(t) = (2t-1) - u(t-2)(t-1)$

$\mathcal{L}[f(t)](s) = 2\mathcal{L}[t] - \mathcal{L}[1] - \mathcal{L}[u(t-2)(t-1)]$

$\mathcal{L}[f(t)](s) = \frac{2}{s^2} - \frac{1}{s} - \mathcal{L}[u(t-2)t] + \mathcal{L}[u(t-2)]$

$\mathcal{L}[f(t)](s) = \frac{2}{s^2} - \frac{1}{s} - \mathcal{L}[u(t-2)(t-2+2)] + \frac{e^{-2s}}{s}$

$\mathcal{L}[f(t)](s) = \frac{2}{s^2} - \frac{1}{s} - e^{-2s}\left(\frac{1}{s^2} + \frac{2s}{s^2}\right) + \frac{e^{-2s}}{s}$

15) Express the given function in terms of unit step functions, and then find its Laplace transform.

$$f(t) = \begin{cases} -t & 0 \leq t < 2 \\ t-4 & 2 \leq t < 3 \\ 1 & t \geq 3 \end{cases}$$

Solution

Using the Heaviside function, we write:
$f(t) = -t\,(u(t-0) - u(t-2)) + (t-4)\,(u(t-2) - u(t-3)) + 1\,(u(t-3) - u(t-\infty))$

Simplifying, $f(t) = -t + u(t-2)(2t-4) - u(t-3)(t-5)$

$\mathcal{L}[f(t)](s) = -\mathcal{L}[t] + 2\mathcal{L}[u(t-2)(t-2)] - \mathcal{L}[u(t-3)(t-5)]$

$\mathcal{L}[f(t)](s) = -\dfrac{1}{s^2} + \dfrac{2e^{-2s}}{s} - e^{-3s}\mathcal{L}[t+3] + \dfrac{5e^{-3s}}{s}$

$\mathcal{L}[f(t)](s) = -\dfrac{1}{s^2} + \dfrac{2e^{-2s} + 5e^{-3s}}{s} - e^{-3s}\left(\dfrac{1}{s^2} + \dfrac{3}{5}\right)$

16) Find the inverse Laplace transforms of the following functions both in terms of step functions and in terms of piecewise defined functions. $H(s) = \dfrac{e^{-s}}{s^3} + \dfrac{e^{-2s}}{s^2}$

Solution

$\mathcal{L}^{-1}[H(s)](t) = \mathcal{L}^{-1}\left[\dfrac{e^{-s}}{s^3}\right] + \mathcal{L}^{-1}\left[\dfrac{e^{-2s}}{s^2}\right]$

$\mathcal{L}^{-1}[H(s)](t) = u(t-1)\,\mathcal{L}^{-1}\left[\dfrac{1}{s^3}\right] + u(t-2)\,\mathcal{L}^{-1}\left[\dfrac{1}{s^2}\right]$

$\mathcal{L}^{-1}[H(s)](t) = u(t-1)\,\dfrac{(t-1)^2}{2} + u(t-2)(t-2) = h(t)$

$h(t)$ expressed in piecewise notation:

$$h(t) = \begin{cases} 0 & t < 1 \\ \frac{t^2}{2} & 1 \leq t < 2 \\ t + \frac{t^2}{s} & t \geq 2 \end{cases}$$

17) Find the Laplace transform of the following function.

$$f(t) = \int_0^t \sin(a\tau)\cos(b(t-\tau))\,d\tau$$

Solution

$\displaystyle\int_0^t \sin(a\tau)\cos(b(t-\tau))\,d\tau = \sin(at) * \cos(bt)$

Using the transform of a convolution: $\mathcal{L}[f(t)](s) = \mathcal{L}[\sin(at)] \cdot \mathcal{L}[\cos(bt)]$

$\mathcal{L}[f(t)](s) = \left(\frac{a}{s^2+a^2}\right)\left(\frac{s}{s^2+b^2}\right) = \frac{as}{s^4+s^2(a^2+b^2)+a^2b^2}$

18) Find the Laplace transform of the following function.

$$g(t) = e^t \int_0^t \sin(\omega\tau)\cos(\omega(t-\tau))\,d\tau$$

Solution

$e^t \int_0^t \sin(\omega\tau)\cos(\omega(t-\tau))\,d\tau = e^t(\sin(\omega t) * \cos(\omega t))$

Using the transform of a convolution: $\mathcal{L}[g(t)](s) = \mathcal{L}[e^t(\sin(\omega t) * \cos(\omega t))] = \mathcal{L}[\sin(\omega t) * \cos(\omega t)](s-1)$

$\mathcal{L}[g(t)](s) = \mathcal{L}[\sin(\omega t)](s-1) \cdot \mathcal{L}[\cos(\omega t)](s-1)$

$\mathcal{L}[g(t)](s) = \left(\frac{\omega}{(s-1)^2+\omega^2}\right)\left(\frac{s-1}{(s-1)^2+\omega^2}\right) = \frac{\omega s-\omega}{\left((s-1)^2+\omega^2\right)^2}$

19) Find the Laplace transform of the following function.

$$f(t) = \int_0^t \tau(t-\tau)\sin(\omega\tau)\cos(\omega(t-\tau))\,d\tau$$

Solution

$\int_0^t \tau(t-\tau)\sin(\omega\tau)\cos(\omega(t-\tau))\,d\tau = t\sin(\omega t) * \cos(\omega t)$

Using the transform of a convolution: $\mathcal{L}[f(t)](s) = \mathcal{L}[t\sin(\omega t)] \cdot \mathcal{L}[\cos(\omega t)] = G(s) \cdot H(s)$

Let $g(t) = t\sin(\omega t)$ and $h(t) = t\cos(\omega t)$ so that $\mathcal{L}[g(t)] = G(s)$ and $\mathcal{L}[h(t)] = H(s)$

$\mathcal{L}[g'(t)] = sG(s) - g(0)$

$\mathcal{L}[\sin(\omega t) + t\omega\cos(\omega t)] = sG(s)$

$\frac{\omega}{s^2+\omega^2} + \omega H(s) = sG(s)$

Rearrange this to obtain: $sG(s) - \omega H(s) = \frac{\omega}{s^2+\omega^2}$

We also have: $\mathcal{L}[h'(t)] = sH(s) - h(0)$

$\mathcal{L}[\cos(\omega t)] - \omega\mathcal{L}[H] = sH(s)$

$\frac{s}{s^2+\omega^2} - \omega G(s) = sH(s)$

Rearrange this for $sH(s) + \omega G(s) = \frac{s}{s^2+\omega^2}$

We now have the system:

$sG(s) - \omega H(s) = \frac{\omega}{s^2+\omega^2}$

$sH(s) + \omega G(s) = \frac{s}{s^2+\omega^2}$

This can be solved to find: $G(s) = \dfrac{2s\omega}{s^2+\omega^2}$ and $H(s) = \dfrac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}$

So the final solution is $\mathcal{L}[f(t)](s) = G(s) \cdot H(s) = \dfrac{2s^3\omega - 2s\omega^3}{\left(s^2+\omega^2\right)^4}$

20) Find the inverse Laplace transforms of the following functions both in terms of step functions and in terms of piecewise defined functions. $H(s) = \dfrac{e^{-\pi s}(1-2s)}{s^2+4s+5}$

Solution

$H(s) = \dfrac{e^{-\pi s}(1-2s)}{s^2+4s+5} = \dfrac{e^{-\pi s}(1-2s)}{(s+2)^2+1}$

$\mathcal{L}^{-1}[H(s)](t) = u(t-\pi)\left(\mathcal{L}^{-1}\left[\dfrac{1}{(s+2)^2+1}\right] - 2\mathcal{L}^{-1}\left[\dfrac{s}{(s+2)^2+1}\right)\right]$

$\mathcal{L}^{-1}[H(s)](t) = u(t-\pi)\left(e^{-2t}\mathcal{L}^{-1}\left[\dfrac{1}{s^2+1}\right] - 2e^{-2t}\mathcal{L}^{-1}\left[\dfrac{s}{s^2+1}\right]\right)$

$\mathcal{L}^{-1}[H(s)](t) = u(t-\pi)\left(e^{-2t}\sin(t) - e^{-2t}\cos(t)\right)$

$\mathcal{L}^{-1}[H(s)](t) = e^{-2t}\sin(t)\,u(t-\pi) - 2e^{-2t}\cos(t)\,u(t-\pi)$

21) Solve the following equation using the Laplace transform. $y'' - y' - 2y = 5\sin x$, $y(0) = 1$, $y'(0) = -1$

Solution

Let $\mathcal{L}[y](s) = Y(s)$

$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] = 5\mathcal{L}[\sin(x)]$

Using the transform of derivatives we find:
$s^2Y(s) - sY(s) - 2Y(s) - s + 2 = \dfrac{5}{s^2+1}$

$Y(s)\left(s^2-s-2\right) - s + 2 = \dfrac{5}{s^2+1}$, $Y(s) = \dfrac{5}{(s-2)(s+1)(s^2+1)} + \dfrac{1}{s+1}$

Using partial fraction decomposition to rewrite the first term on the RHS:
$\dfrac{5}{(s-2)(s+1)(s^2+1)} = \dfrac{A}{s-2} + \dfrac{B}{s+1} + \dfrac{Cs+D}{s^2+1}$

Solving, $A = \frac{1}{3}, B = -\frac{5}{6}, C = \frac{1}{2}, D = -\frac{3}{2}$

So now $\dfrac{5}{(s-2)(s+1)(s^2+1)} = \dfrac{1}{3(s-2)} - \dfrac{5}{6(s+1)} + \dfrac{s-3}{2(s^2+1)}$

$y(t) = \dfrac{1}{3}\mathcal{L}^{-1}\left[\dfrac{1}{s-2}\right] - \dfrac{5}{6}\mathcal{L}^{-1}\left[\dfrac{1}{s+1}\right] + \dfrac{1}{2}\mathcal{L}^{-1}\left[\dfrac{s}{s^2+1}\right] - \dfrac{3}{2}\mathcal{L}^{-1}\left[\dfrac{1}{s^2+1}\right]$

$y(t) = \dfrac{1}{3}e^{2x} + \dfrac{1}{6}e^{-x} + \dfrac{1}{2}\cos(x) - \dfrac{3}{2}\sin(x)$

22) Solve the initial value problem. $y'' + 9y = u(t-2\pi)\sin t$, where $y(0) = 1$, and $y'(0) = 0$.

Solution

Let $\mathcal{L}[y](s) = Y(s)$

$\mathcal{L}[y''] + 9\mathcal{L}[y] = \mathcal{L}[u(t-2\pi)\sin(t)] = \mathcal{L}[u(t-2\pi)\sin(t-2\pi+2\pi)]$

Using the transform of derivatives:
$s^2Y(s) + 9sY(s) - s - 9 = e^{-2\pi s}\left(\frac{1}{s^2+1}\right)$

$Y(s) = \dfrac{e^{-2\pi s}}{s(s+9)(s^2+1)} + \dfrac{1}{s}$, $y(t) = \mathcal{L}^{-1}\left[\dfrac{e^{-2\pi s}}{s(s+9)(s^2+1)} + \dfrac{1}{s}\right]$

23) Solve the integral equation.

$$y(t) = \sin t - 2\int_0^t \cos(t-\tau)\,y(\tau)\,d\tau$$

Solution

Let $\mathcal{L}[y](s) = Y(s)$

$\mathcal{L}[y] = \mathcal{L}[\sin(t)] - 2\mathcal{L}\left[\int_0^t \cos(t-\tau)\,y(\tau)\,d\tau\right]$

Using the transform of an integral:
$Y(s) = \dfrac{1}{s^2+1} - 2\left(\mathcal{L}[\cos(t)]\,\mathcal{L}[y]\right)$

$Y(s) = \dfrac{1}{s^2+1} - 2\left(\dfrac{s}{s^2+1}\right)(Y(s))$

$Y(s) + \dfrac{2s}{s^2+1}Y(s) = \dfrac{1}{s^2+1}$, $Y(s) = \dfrac{1}{(s+1)^2}$

$y(t) = \mathcal{L}^{-1}\left[\dfrac{1}{(s+1)^2}\right] = e^{-t}\mathcal{L}^{-1}\left[\dfrac{1}{s^2}\right] = e^{-t}t$

24) Solve the integral equation.

$$y'(t) = t + \int_0^t y(\tau)\cos(t-\tau)\,d\tau, \qquad y(0) = 4$$

Solution

Let $Y(s) = \mathcal{L}[y]$

$\mathcal{L}[y'] = \mathcal{L}[t] + \mathcal{L}[y * \cos(t)]$

Using the transform of derivatives and convolutions:
$sY(s) - y(0) = \frac{1}{s^2} + Y(s)\left(\frac{s}{s^2+1}\right)$

Rearrange for: $sY(s) - Y(s)\left(\dfrac{s}{s^2+1}\right) = \dfrac{1}{s^2} + 4$

Solve for $Y(s)$: $Y(s) = \dfrac{s^2+1}{s^5} + \dfrac{4s^2+4}{s^3} = \dfrac{1}{s^3} + \dfrac{1}{s^5} + \dfrac{4}{s} + \dfrac{4}{s^3}$

So $y(t) = \mathcal{L}^{-1}\left[\dfrac{1}{s^3}\right] + \mathcal{L}^{-1}\left[\dfrac{1}{s^5}\right] + 4\mathcal{L}^{-1}\left[\dfrac{1}{s}\right] + 4\mathcal{L}^{-1}\left[\dfrac{1}{s^3}\right]$

$y\left(t\right)=\dfrac{5t^2}{2}+\dfrac{t^4}{24}+4$

25) Evaluate the integral (Hint: Use the convolution theorem).

$$\int_0^t \left(t-\tau\right)^{13}\tau^7 d\tau$$

Solution

$\int_0^t \left(t-\tau\right)^{13}\tau^7 d\tau = t^{13} * t^7$

$\mathcal{L}\left[f\left(t\right)\right]=\mathcal{L}\left[t^{13}\right]\cdot\mathcal{L}\left[t^7\right]$

$\mathcal{L}\left[f\left(t\right)\right]=\left(\dfrac{13!}{s^{14}}\right)\left(\dfrac{7!}{s^8}\right)=\dfrac{13!7!}{s^{22}}$

$\mathcal{L}^{-1}\left[\dfrac{13!7!}{s^{22}}\right]=13!\,7!\,\dfrac{t^{21}}{21!}$

26) Solve the initial value problem. $y''+3y'+2y=6e^{2t}+2\delta\left(t-1\right)$, $y(0)=2$, $y'(0)=-6$.

Solution

Let $\mathcal{L}\left[y\right]=Y\left(s\right)$

$\mathcal{L}\left[y''\right]+3\mathcal{L}\left[y'\right]+2\mathcal{L}\left[y\right]=6\mathcal{L}\left[e^{2t}\right]+2\mathcal{L}\left[\delta\left(t-1\right)\right]$

$s^2Y\left(s\right)=3sY\left(s\right)+2Y\left(s\right)-2s=\dfrac{6}{s-2}+2e^{-s}$

$Y\left(s\right)\left(s^2+3s+2\right)=\dfrac{6}{s-2}+2e^{-s}+2s$

Solving for $Y\left(s\right)$: $Y\left(s\right)=\dfrac{6}{\left(s-2\right)\left(s+2\right)\left(s+1\right)}+\dfrac{2e^{-s}}{\left(s+2\right)\left(s+1\right)}+\dfrac{2s}{\left(s+2\right)\left(s+1\right)}$

Using partial fraction decomposition, we can rewrite all three terms.

Term 1: $\dfrac{1}{\left(s-2\right)\left(s+2\right)\left(s+1\right)}=\dfrac{A}{s-2}+\dfrac{B}{s+2}+\dfrac{C}{s+1}$
$=\dfrac{1}{12\left(s-2\right)}+\dfrac{1}{4\left(s+2\right)}-\dfrac{1}{3\left(s+1\right)}$

Term 2: $\dfrac{1}{\left(s+2\right)\left(s+1\right)}=\dfrac{A}{s+2}+\dfrac{1}{s+1}=-\dfrac{1}{s+2}+\dfrac{1}{s+1}$

Term 3: $\dfrac{s}{\left(s+2\right)\left(s+1\right)}=\dfrac{A}{s+2}+\dfrac{B}{s+1}=\dfrac{2}{s+2}-\dfrac{1}{s+1}$

Recombining, we can now see

$y\left(t\right)=6\mathcal{L}^{-1}[\dfrac{1}{12\left(s-2\right)}+\dfrac{1}{4\left(s+2\right)}-\dfrac{1}{3\left(s+1\right)}+2\mathcal{L}^{-1}\left[e^{-s}\left(-\dfrac{1}{s+2}+\dfrac{1}{s+1}\right)\right]+2\mathcal{L}^{-1}\left[\dfrac{2}{s+2}-\dfrac{1}{s+1}\right]$

$y\left(t\right)=6\left(\dfrac{1}{12}e^{2t}+\dfrac{1}{4}e^{-2t}-\dfrac{1}{3}e^{-t}\right)+2u\left(t-1\right)\left(-e^{-2t}+e^{-t}\right)+4e^{-2t}-2e^{-t}$

$y\left(t\right)=\dfrac{e^{2t}}{2}+\dfrac{11}{2}e^{-2t}-4e^{-t}+2u\left(t-1\right)\left(e^{-t}-e^{-2t}\right)$

27) Solve the initial value problem. $y''+4y=\sin t+\delta\left(t-\pi/2\right)$, $y\left(0\right)=0$, $y'\left(0\right)=2$

Solution

First we compute the Laplace transform of the differential equation.

$$\left(s^2+4\right)Y\left(s\right)-2=\frac{1}{s^2+1}+e^{-s\pi/2}$$

$$Y\left(s\right)=\frac{1}{\left(s^2+1\right)\left(s^2+4\right)}+\frac{e^{-s\pi/2}}{s^2+4}+\frac{2}{s^2+4}$$

$$\mathcal{L}^{-1}\left(Y\left(s\right)\right)=\frac{1}{2}\cdot\frac{5}{3}\sin\left(2t\right)+\frac{1}{2}u\left(t-\pi/2\right)\sin\left(2\left(t-\pi/2\right)\right)+\frac{1}{3}\sin t$$

$$y\left(t\right)=\frac{5}{6}\sin\left(2t\right)-\frac{1}{2}u\left(t-\pi/2\right)\sin\left(2t\right)+\frac{1}{3}\sin t$$

28) Solve the system.

$$\mathbf{x}'=\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}\mathbf{x}$$

Solution

First we find the eigenvalues of the matrix to be $\lambda_1=-2$, $\lambda_2=3$. The corresponding eigenvectors are $\mathbf{v}_1=\begin{pmatrix}1\\1\end{pmatrix}$, $\mathbf{v}_2=\begin{pmatrix}-4\\1\end{pmatrix}$. Thus the general solution is

$$\mathbf{x}=c_1e^{-2t}\begin{pmatrix}1\\1\end{pmatrix}+c_2e^{3t}\begin{pmatrix}-4\\1\end{pmatrix}$$

29) Solve the system.

$$\mathbf{y}'=\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\mathbf{y}$$

Solution

First we find the eigenvalues of the matrix to be $\lambda_1=1$, $\lambda_2=2$. The corresponding eigenvectors are $\mathbf{v}_1=\begin{pmatrix}1\\1\end{pmatrix}$, $\mathbf{v}_2=\begin{pmatrix}3\\2\end{pmatrix}$. Thus the general solution is

$$\mathbf{y}=c_1e^{x}\begin{pmatrix}1\\1\end{pmatrix}+c_2e^{2x}\begin{pmatrix}3\\2\end{pmatrix}$$

30) Solve the initial value problem

$$\mathbf{x}'=\begin{pmatrix} -2 & 1 \\ -2 & -5 \end{pmatrix}\mathbf{x},\qquad \mathbf{x}(0)=\begin{pmatrix}5\\2\end{pmatrix}$$

Solution

First we find the eigenvalues of the matrix to be $\lambda_1=-3$, $\lambda_2=-4$. The corresponding eigenvectors are $\mathbf{v}_1=\begin{pmatrix}1\\-1\end{pmatrix}$, $\mathbf{v}_2=\begin{pmatrix}-1\\2\end{pmatrix}$. Thus the general solution is

$$\mathbf{x}=c_1e^{-3t}\begin{pmatrix}1\\-1\end{pmatrix}+c_2e^{-4t}\begin{pmatrix}-1\\2\end{pmatrix}$$

Now, plugging in the initial condition,

$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Thus $c_1 = 12$, $c_2 = 7$. So then the solution is

$$\mathbf{x} = 12e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 7e^{-4t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$