Mathematics 100 Quiz 3 Review Package – Solutions

UBC Engineering Undergraduate Society

Attempt questions to the best of your ability. This review package consists of 11 pages, including 1 cover page and 17 questions. The questions are meant to be the level of a real examination or slightly above, in order to prepare you for the real exam. Material from lectures and from the relevant textbook sections is examinable, and the problems for this package were chosen with that in mind, as well as considerations based on past examination question difficulty and style. Problems are ranked in difficulty as (•) for easy, (••) for medium, and (•••) for difficult. Note that sometimes difficulty can be subjective, so do not be discouraged if you are stuck on a (•) problem.

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Some of the problems in this package were not created by the EUS. Those problems originated from one of the following sources:

- Schuam’s Outline of Calculus 2 ed; Ayres Jr., Frank
- Calculus – Early Transcendentals 7 ed; Stewart, James
- Calculus – 3 ed; Spivak, Michael
- Calculus Volume 1 2 ed; Apostol, Tom

All solutions prepared by the EUS.

Good Luck!
1. Given \( x^2y - xy^2 + x^2 + y^2 = 0 \),

(a) Regarding \( y \) as a function of \( x \), compute \( \frac{dy}{dx} \).

(b) Compute a linear approximation to this graph at the point \((1, -1)\).

Solution:

(a) Use implicit differentiation: Let’s find the derivative of each term separately, then add them all together. First the derivative of the first summand:

\[
[x^2y]' = (2x)(y) + (x^2)(y') = 2xy + y'x^2
\]

Derivative of the second summand:

\[
[xy^2]' = (1)(y^2) + (x)(2yy') = y^2 + 2xyy'
\]

Derivative of the third summand:

\[
[x^2]' = 2x
\]

Derivative of the last summand:

\[
[y^2]' = 2yy'
\]

Adding all of the summands together:

\[
0 = 2xy + x^2y' - y^2 - 2xyy' + 2x + 2yy'
\]

Solving for \( y' \):

\[
y' = \frac{2xy - y^2 + 2x}{-x^2 + 2xy - 2y}
\]

(b) We must evaluate \( y' \) at the point \((1, -1)\). It is \( y'(1, -1) = 1 \) Then the linear approximation is

\[
L(x) = -1 + 1(x - 1) = x - 2
\]

2. Considering \( y \) as a function of \( x \), compute \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \).

\( x^2 - xy + y^2 = 3 \)

Solution: Use implicit differentiation to find both the first and second derivative.

\[
\frac{d}{dx} [x^2 - xy + y^2] = \frac{d}{dx} [3]
\]
\[ \begin{align*}
2x - (y + xy') + 2yy' &= 0 \\
2x - y - xy' + 2yy' &= 0 \\
2x - y &= xy' = 2yy' \\
y' &= \frac{2x - y}{x - 2y}
\end{align*} \]

Now for the second derivative: Note that will have to plug the first derivative into the formula for the second derivative.

\[ \begin{align*}
y'' &= \frac{d}{dx} [y'] \\
&= \frac{2x - y}{x - 2y} \\
&= \frac{(2 - y')(x - 2y) - (2x - y)(1 - 2y')}{(x - 2y)^2} \\
&= \frac{2x - 4y - xy' + 2yy' - (2x - 4xy' - y + 2yy')}{(x - 2y)^2} \\
&= \frac{-3y + 3xy'}{(x - 2y)^2} \\
&= \frac{3(xy' - y)}{(x - 2y)^2} \\
&= \frac{3 \left( x \left( \frac{2x - y}{x - 2y} \right) - y \right)}{(x - 2y)^2} \\
&= \frac{3 \left( \frac{2x - y}{x - 2y} - y \right)}{(x - 2y)^2} \\
&= \frac{3 \left( \frac{2x - y}{x - 2y} - \frac{y(x - 2y)}{x - 2y} \right)}{(x - 2y)^2} \\
y'' &= \frac{6(x^2 + y^2 - xy)}{(x - 2y)^3}
\end{align*} \]

(**) 3. Prove that the two curves intersect at right angles at the origin.

(i) \[ 5y - 2x + y^3 - x^2y = 0 \]
(ii) \[ 2y + 5x + x^4 - x^3y^2 = 0 \]

**Solution:** Since plugging \((0,0)\) into both (i) and (ii) satisfies the equation, we know that both curves pass through the origin.
(i)
\[
\frac{d}{dx}[5y - 2x + y^3 - x^2y] = \frac{d}{dx}[0]
\]
\[
5y' - 2 + 3y^2y' - (2xy + x^2y') = 0
\]
\[
5y' + 3y^2y' - x^2y' - 2 - 2xy = 0
\]
\[
y'(5 + 3y^2 - x^2) = 2 + 2xy
\]
\[
y' = \frac{2 + 2xy}{5 + 3y^2 - x^2}
\]
Evaluating at \(x = 0\), we have
\[
y'(0) = \frac{2 + 2 \cdot 0 \cdot 0}{5 + 3 \cdot 0 - 0} = \frac{2}{5}
\]

(ii)
\[
\frac{d}{dx}[2y + 5x + x^4 - x^3y^2] = \frac{d}{dx}[0]
\]
\[
2y' + 5 + 4x^3 - (3x^2y^2 + x^3 \cdot 2yy') = 0
\]
\[
2y' - 2x^3yy' + 5 + 4x^3 - 3x^2y^2 = 0
\]
\[
y'(2 - 2x^3y) = \frac{3x^2y^2 - 5 - 4x^3}{2 - 2x^3y}
\]
\[
y' = \frac{3x^2y^2 - 5 - 4x^3}{2 - 2x^3y}
\]
Evaluating at \(x = 0\) we have
\[
y'(0) = \frac{-5}{2}
\]
Both equations pass through the origin and each equation’s slope at the origin is the negative reciprocal of the other, therefore they are perpendicular.

(**) 4. If \(\sin y + \cos x = 1\), compute \(y''\) by using implicit differentiation.

**Solution:**
\[
\frac{d}{dx}[1] = \frac{d}{dx}[\sin y] + \frac{d}{dx}[\cos x]
\]
\[
0 = \cos(y) \cdot y' - \sin(x)
\]
\[
y' = \frac{\sin(x)}{\cos(y)}
\]
Now use the quotient rule to get the second derivative
\[ y'' = \frac{(\sin(x))'(\cos(y)) - (\sin(x))(\cos(y))'}{\cos^2(y)} \]
\[ y'' = \frac{\cos(x) \cdot \cos(y) - \sin(x) \cdot (-\sin(y) \cdot y')}{\cos^2(y)} \]
\[ y'' = \frac{\cos(x) \cdot \cos(y) + \sin(x) \sin(y) \cdot \frac{\sin(x)}{\cos(y)}}{\cos^2(y)} \]
\[ y'' = \frac{\cos(x) + \frac{\sin^2(x) \sin(y)}{\cos(y)}}{\cos^2(y)} \]
\[ y'' = \frac{\cos(x) + \frac{\sin^2(x) \tan(y)}{\cos^2(y)}}{\cos^2(y)} \]

5. Regarding \( y \) as a function of \( x \), compute \( y' \):

\[ x \cos y = \sin(x + y) \]

**Solution:** First, use the identity \( \sin(u \pm v) = \sin(u) \cos(v) \pm \cos(u) \sin(v) \), then use the product rule for each of the three terms.

\[ x \cos(y) = \sin(x) \cos(y) + \cos(x) \sin(y) \]

Now differentiating each term. First we differentiate the left hand side:

\[ \frac{dy}{dx} [x \cos(y)] = \cos(y) + x(-\sin(y))y' \]
\[ = \cos(y) - xy' \sin(y) \]

Now we differentiate the first summand on the right hand side:

\[ \frac{dy}{dx} [\sin(x) \cos(y)] = \cos(x) \cos(y) + \sin(x)(-\sin(y) \cdot y') \]
\[ = \cos(x) \cos(y) - \sin(x) \sin(y)y' \]

Now we differentiate the second summand on the right hand side:

\[ \frac{dy}{dx} [\cos(x) \sin(y)] = -\sin(x) \sin(y) + \cos(x)(\cos(y)y') \]
\[ = \cos(x) \cos(y)y' - \sin(x) \sin(y) \]

Adding it all up

\[ \cos(y) - xy' \sin(y) = \cos(x) \cos(y) - \sin(x) \sin(y)y' + \cos(x) \cos(y)y' - \sin(x) \sin(y) \]

Rearranging all of the \( y' \) terms on one side:

\[ \cos(y) - \cos(x) \cos(y) + \sin(x) \sin(y) = \cos(x) \cos(y)y' - \sin(x) \sin(y)y' + x \sin(y)y' \]

Finally solving for \( y' \):

\[ y' = \frac{\cos(y) - \cos(x) \cos(y) + \sin(x) \sin(y)}{\cos(x) \cos(y) - \sin(x) \sin(y) + x \sin(y)} \]
6. Differentiate the following function. Hint: Use logarithmic differentiation.

\[ x(t) = \frac{6(1 + t^2)(t^3 - t)^2}{(4t)^{3/2}\sqrt{t + 5t^2}} + \frac{\sqrt{1 + t^2}}{t + \sqrt{1 + t^2}} \]

**Solution:** Define

\[ f(t) = \frac{6(1 + t^2)(t^3 - t)^2}{(4t)^{3/2}\sqrt{t + 5t^2}} \]
\[ g(t) = \frac{\sqrt{1 + t^2}}{t + \sqrt{1 + t^2}} \]

Then we take the logarithm of both sides of \( f \):

\[ \log f = \log 6 + \log(1 + t^2) + 2 \log(t^3 - t) - \frac{3}{2} \log(4t) - \frac{1}{2} \log(t + 5t^2) \]

Differentiating both sides with respect to \( t \):

\[ \frac{f'(t)}{f(t)} = \frac{2t}{1 + t^2} + 2 \cdot \frac{3t^2 - 1}{t^3 - t} - \frac{3}{2t} - \frac{1 + 10t}{2t + 10t^2} \]

Now we take the logarithm of \( g \)

\[ \log g = \frac{1}{2} \log(1 + 2t) - \log(t + \sqrt{1 + t^2}) \]

Now differentiating both sides with respect to \( t \):

\[ \frac{g'(t)}{g(t)} = \frac{1}{1 + 2t} - \frac{1}{\sqrt{1 + t^2}} \]

Now that we have \( f' \) and \( g' \), we can add them together to find \( x' \):

\[ \frac{dx}{dt} = \frac{df}{dt} + \frac{dg}{dt} \]

\[ \frac{dx}{dt} = \frac{6(1 + t^2)(t^3 - t)^2}{(4t)^{3/2}\sqrt{t + 5t^2}} \left( \frac{2t}{1 + t^2} + 2 \cdot \frac{3t^2 - 1}{t^3 - t} - \frac{3}{2t} - \frac{1 + 10t}{2t + 10t^2} \right) + \frac{\sqrt{1 + t^2}}{t + \sqrt{1 + t^2}} \left( \frac{1}{1 + 2t} - \frac{1}{\sqrt{1 + t^2}} \right) \]

7. Differentiate the following function. \( y = 2^{\tan x} \)

**Solution:** First take the logarithm of both sides:

\[ \log(y) = \log(2^{\tan(x)}) \]

Differentiate both sides:

\[ \frac{dy}{dx} \log(y) = \frac{dy}{dx} \left( \tan(x) \cdot \log(2) \right) \]
\[ \frac{y'}{y} = \log(2) \cdot \sec^2(x) \]
Rearrange to isolate $y'$:

$$y' = y \cdot \log(2) \cdot \sec^2(x)$$

Plugging back in $y$:

$$y' = 2^{\tan(x)} \cdot \log(2) \cdot \sec^2(x)$$

8. Differentiate the following function. $y = (\tan x)^{1-x^2}$

**Solution:**

First take the logarithm of both sides:

$$\log(y) = \log(\tan(x)) \cdot (1 - x^2)$$

Differentiating both sides:

$$\frac{y'}{y} = [\log(\tan(x)) \cdot (1 - x^2)]'$$

$$= \left( \frac{\sec^2(x)}{\tan(x)} \right) (1 - x^2) + (\log(\tan(x)))(-2x)$$

Isolating $y'$:

$$y' = y \left( \frac{(1 - x^2) \sec^2(x)}{\tan(x)} - 2x \log(\tan(x)) \right)$$

$$y' = (\tan(x))^{1-x^2} \left( \frac{(1 - x^2) \sec^2(x)}{\tan(x)} - 2x \log(\tan(x)) \right)$$

9. Differentiate the following function. $y = (\arcsin(3x))^{\sqrt{e^{3x}+1}}$

**Solution:**

Taking the logarithm of both sides, and then differentiating:

$$\frac{y'}{y} = \frac{3\sqrt{e^{3x}+1}}{\arcsin(3x)\sqrt{1-9x^2}} + \log(\arcsin(3x)) \frac{e^{3x}}{2\sqrt{e^{3x}+1}}$$

Isolating $y'$:

$$\frac{dy}{dx} = (\arcsin(3x))^{\sqrt{e^{3x}+1}} \cdot \left( \frac{3\sqrt{e^{3x}+1}}{\arcsin(3x)\sqrt{1-9x^2}} + \log(\arcsin(3x)) \frac{3e^{3x}}{2\sqrt{e^{3x}+1}} \right)$$

10. Compute a linear approximation to $f(x)$ at the point $x = \pi$.

$$f(x) = x^{x^{\sin x}}$$
**Solution:** Take the logarithm of both sides twice:

\[ \log \log f = \sin x \log x + \log \log x. \]

Differentiating both sides:

\[ \frac{f'}{f \log f} = \frac{\sin x}{x} + \cos x \log x + \frac{1}{x \log x} \]

Solving for \( f' \):

\[ f' = x^{\sin x} \cdot (x^{\sin x} \log x) \cdot \left( \frac{\sin x}{x} + \cos x \log x + \frac{1}{x \log x} \right) \]

Now evaluating \( f'(\pi) \):

\[ f'(\pi) = 1 - \pi (\log \pi)^2 \]

Plugging that value into the linear approximation formula

\[ L(x) = f(\pi) + f'(\pi)(x - \pi) \]

\[ = \pi + (1 - \pi (\log \pi)^2)(x - \pi) \]

(\( \ast \)) 11. Differentiate the following function. \( y = \arctan \left( \frac{3}{x} \right) \)

**Solution:** Apply the chain rule:

\[ y' = \frac{1}{1 + \left( \frac{3}{x} \right)^2} \cdot -\frac{3}{x^2} \]

\[ = -\frac{3}{x^2(1 + \frac{9}{x^2})} \]

\[ = -\frac{3}{x^2 + 9} \]

(\( \ast \ast \)) 12. Differentiate the following function. \( y = x^2 \arccos \left( \frac{2}{x} \right) \)

**Solution:** First we find the derivative of \( \arccos \left( \frac{2}{x} \right) \), then apply the product rule to the entire function.

\[ \left[ \arccos \left( \frac{2}{x} \right) \right]' = -\frac{1}{\sqrt{1 - \left( \frac{2}{x} \right)^2}} \cdot -\frac{2}{x^2} \]

\[ = \frac{2}{x^2 \sqrt{1 - \frac{4}{x^2}}} \]

\[ = \frac{2}{x \sqrt{x^2 - 4}} \]
Now we differentiate the entire function:

\[
y' = \left[ x^2 \arccos\left( \frac{2}{x} \right) \right]' = (2x) \left[ \arccos\left( \frac{2}{x} \right) \right] + \left( x^2 \right) \left( \frac{2}{x\sqrt{x^2 - 4}} \right)
\]

\[
= 2x \arccos\left( \frac{2}{x} \right) + \frac{2x}{\sqrt{x^2 - 4}}
\]

(**) 13. Differentiate the following function. \( y = \arcsin(e^x \cdot \tan x) \)

**Solution:**

\[
y' = \frac{e^x(\tan x + \sec^2 x)}{\sqrt{1 - (e^x \tan x)^2}}
\]

(**) 14. Prove that if \( f \) is an invertible, differentiable function, then

\[
(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}
\]

**Hint 1.** Recall that \( f(f^{-1}(x)) = x \)

**Solution:** Consider the expression

\[
f(f^{-1}(x)) = x
\]

and differentiate both sides with respect to \( x \).

\[
f'(f^{-1}(x))(f^{-1})'(x) = 1
\]

Then rearranging, we have

\[
(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}
\]

(***) 15. Suppose \( h \) is a function such that \( h'(x) = \sin^2(\sin(x + 1)) \), and \( h(0) = 3 \). Find

(a) \( (h^{-1})'(3) \),

(b) \( (\beta^{-1})'(3) \), where \( \beta(x) = h(x + 1) \).

**Solution:**

(a) From the formula derived above,

\[
(h^{-1})'(x) = \frac{1}{h'(h^{-1}(x))}
\]
Plugging in $x = 3$ to both sides,

$$(h^{-1})'(3) = \frac{1}{h'(h^{-1}(3))}$$

and since $h(0) = 3$, we know that $h^{-1}(3) = 0$. Thus

$$(h^{-1})'(3) = \frac{1}{h'(h^{-1}(3))} = \frac{1}{h'(0)} = \frac{1}{\sin^2(\sin(1))}$$

(b) Since $\beta(x) = h(x + 1)$, we know that $x = \beta^{-1}(h(x + 1))$. Differentiating both sides:

$$(\beta^{-1})'(h(x + 1)) \cdot h'(x + 1) = 1$$

Rearranging:

$$(\beta^{-1})'(h(x + 1)) = \frac{1}{h'(x + 1)}$$

Plugging in $x = -1$ to both sides:

$$(\beta^{-1})'(h(0)) = \frac{1}{h'(0)}$$

$$(\beta^{-1})'(3) = \frac{1}{\sin^2(\sin(1))}$$

(\text{**}) 16. Find a formula for $(f^{-1})''(x)$.

\textbf{Solution:} Given

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))},$$

we can differentiate both sides with respect to $x$.

$$((f^{-1})'(x))' = \left(\frac{1}{f'(f^{-1}(x))}\right)'$$

$$= -\frac{f''(f^{-1}(x))(f^{-1})'(x)}{(f'(f^{-1}(x)))^2}$$

$$= -\frac{1}{(f'(f^{-1}(x)))^2} \cdot f''(f^{-1}(x)) \cdot (f^{-1})'(x)$$

Plugging in the previous formula for $(f^{-1})'$, we have

$$(f^{-1})''(x) = -\frac{f''(f^{-1}(x))}{(f'(f^{-1}(x)))^3}$$

(\text{**}) 17. Suppose that $f$ is a differentiable and invertible function, and that $f = F'$. Let

$$G(x) = xf^{-1}(x) - F(f^{-1}(x)).$$

Show that $G'(x) = f^{-1}(x)$. 

**Solution:** Differentiating both sides:

\[
G'(x) = f^{-1}(x) + x(f^{-1})'(x) - F'(f^{-1}(x)) \cdot (f^{-1})'(x)
\]

\[
= f^{-1}(x) + x(f^{-1})'(x) - f(f^{-1}(x)) \cdot (f^{-1})'(x)
\]

\[
= f^{-1}(x) + x(f^{-1})'(x) - x(f^{-1})'(x)
\]

\[
= f^{-1}(x)
\]